

Full-analytic frequency-domain gravitational wave forms from eccentric compact binaries to 2PN accuracy

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Key words 04.30.Db, 04.30.-w, 04.25.Nx, 97.80.-d

The article provides full-analytic gravitational wave (GW) forms for eccentric nonspinning compact binaries of arbitrary mass ratio in the time Fourier domain. The semi-analytical property of recent descriptions, i.e. the demand of inverting the higher-order Kepler equation numerically but keeping all other computations analytic, is avoided for the first time.

The article is a completion of a previous one (Tessmer and Schäfer, Phys. Rev. D **82**, 124064 (2010)) to second post-Newtonian (2PN) order in the harmonic GW amplitude and conservative orbital dynamics. A fully analytical inversion formula of the Kepler equation in harmonic coordinates is provided, as well as the analytic time Fourier expansion of trigonometric functions of the eccentric anomaly in terms of sines and cosines of the mean anomaly. Tail terms are not considered.

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1 Introduction

In our previous article [1] – it will be called “paper 1” from now onwards – we provided the Fourier domain gravitational wave (GW) forms including the first post-Newtonian (1PN) corrections to the leading order quadrupolar contribution and 1PN corrections to the Newtonian accurate orbital dynamics in terms of tensor spherical harmonics. In this article, we complete the amplitude to 2PN order in harmonic coordinates and we include 2PN orbital dynamics as well. We regard nonspinning compact binaries with arbitrary mass ratio in quasi-elliptical orbits, where the orbital dynamics is well-known. Earlier works provided the solution to the evolution equations in terms of a Keplerian parameterization, perturbed by their post-Newtonian corrections. The result is the so-called quasi-Keplerian parameterization (QKP), to be found, e.g. in [2, 3, 4], which will be one basic for this article. 2PN corrections to the GW amplitude in the limit of non-spinning binaries are also dealt with in [5] where the wave forms are provided with the help of a quasi-Keplerian parameterization, and recently, including spin, in [6] where the results are given in terms of coordinate velocities. We like to improve the use of those wave forms as we provide them as *pure series of harmonics in the time Fourier domain*, as this is one step closer to data analysis investigations. Let us now briefly summarise why 2PN corrections and the inclusion of eccentric orbits are necessary. Current data analysis investigation showed that, even for circular inspirals, the inclusion of 2PN orbital dynamics is necessary to guarantee a successful detection of the GW signal [4, 7, 8]. Authors of a recent publication [9] investigated certain equal-mass black hole binaries in the millisecond GW frequency range in rotating clusters and stated parameter estimation errors for the initial eccentricity of $\sim 10^{-7}$ for LISA. Therefore we claim that to perform an eccentric GW analysis for LISA is demanded. The GW energy flux in terms of its harmonic constitutive parts, was given to 1PN order, in the extreme mass-ratio regime, for the first time – to our best knowledge – in [10]. First attempts, however, for a time Fourier domain

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(TFD) of eccentric h_+ and h_\times , the two polarizations of the far-zone GW field, have been made in Reference [11] where the periastron advance has been incorporated by hand and the stationary phase approximation has come into use, in [12] and [13], where a spectral analysis for steady-state binaries in the simple case of Newtonian motion and amplitude has been performed.

Recently in [14], the authors furnished Newtonian accurate TFD wave forms as they incorporated the sine and cosine functions of the eccentric anomaly as Fourier–Bessel series. Taking as starting point h_+ and h_\times themselves, having evaluated all appearing scalars (such as, for example, the scalar product of the basic vectors of the plane of the sky with the orbital velocity vector) being evaluated explicitly, would lead to rather complicated relations of rotation angles connecting the orientation of the binary’s orbital plane with respect to the position of the observer (see, e.g. [15], and – including spin precession – [16] for higher orders of h_+ and h_\times) and this is not well-suited for a systematic TFD representation at higher orders.

The problem of nonspinning compact binaries implies the conservation of the direction of the orbital angular momentum \mathbf{h} , thus, it qualifies for a representation with the help of tensor spherical harmonics [17, 18, 19]. Reference [20] provided 1PN accurate spherical tensor components for nonspinning compact binaries in quasi-elliptic orbits (and quasi-hyperbolic ones as well) that we are on to verify in this article. Irrespective of the observer’s orientation, the form the GW signal can be expressed with the help of these spherical tensor components and applied to give the polarizations h_+ and h_\times [21] afterwards with the help of constant rotations.

We provide an analytical expression of the eccentric anomaly u in terms of the mean anomaly \mathcal{M} , and do likewise for the sines and cosines of arbitrary integer multiples of u at full 2PN order. By means of this, we are able to express the spherical tensor components in terms of infinite series in harmonics of the elapsed time. To which order the infinite summations have to be driven will be investigated. Similarly to Reference [22], tail terms occurring through 2PN order will not be discussed.

1.1 Organisation of the paper

The paper is organized in the following way. Section 2 collects the orbital elements of the quasi-Keplerian presentation of the 2PN-accurate orbital motion in harmonic coordinates. Section 3 summarizes the transverse-traceless (TT) projection of the far-zone GW field up to 2PN order corrections to the leading-order quadrupole field. The time derivatives of the source multipole moments are computed using expressions for the particle accelerations known from the literature, and the scalar components are projected out with the help of TT tensors in Section 4. The vital components of the Fourier decomposition, here at 2PN accuracy, of the GW field are summarized in Section 5 and the Fourier domain multipoles, incorporating purely 2PN conservative dynamics, are given structurally in its Subsection 5.5. The full explicit expressions can be found in Appendix D.

1.2 Notation

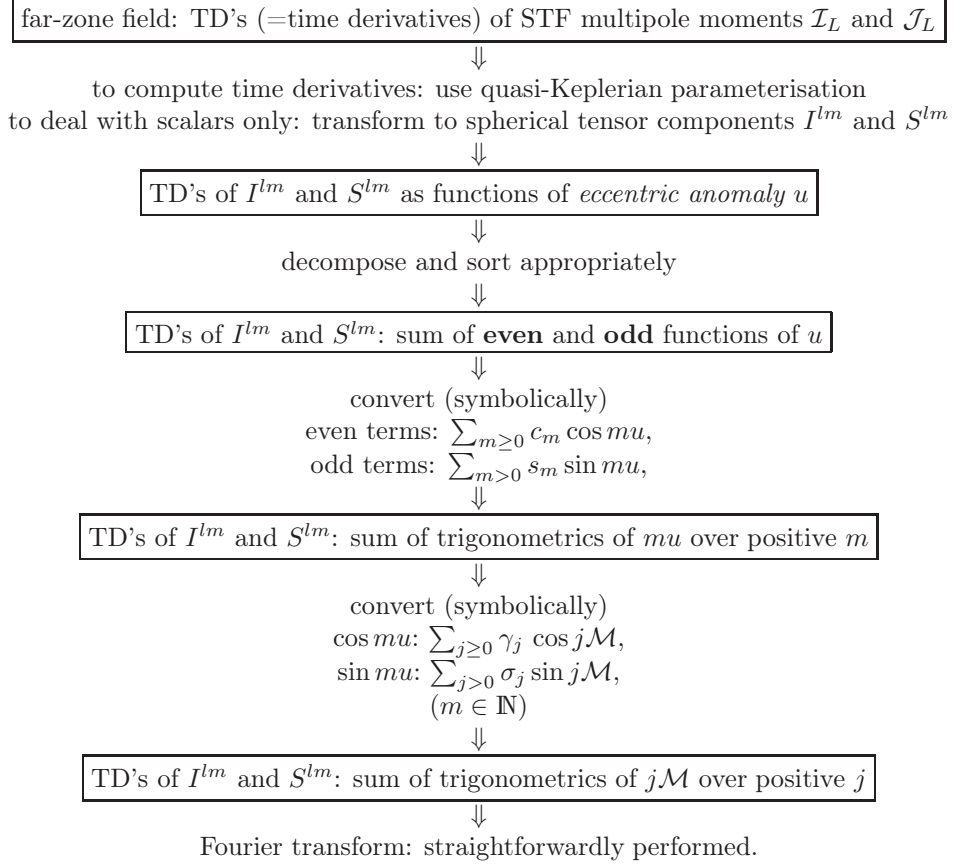
An overview of the quantities which are in use in this paper is given in form of a table of their associated abbreviations and scalings.

u :eccentric anomaly
\mathcal{M} :mean anomaly
\mathbf{h} :orbital vectorial angular momentum, $h := \mathbf{h} $
xPN:x th post Newtonian
rradial separation of the binary components
e_r“radial eccentricity”
e_t“time eccentricity”
e_ϕ“phase eccentricity”
a_rsemimajor axis
ϕorbital phase
vtrue anomaly
Φtotal phase elapsed from a periastron to periastron time
\mathcal{N}mean motion, $\mathcal{N} := \frac{2\pi}{P}$ where P is the time from periastron to periastron
\mathcal{K}periastron advance parameter, $\mathcal{K} := \frac{\Phi - 2\pi}{2\pi}$
η	symmetric mass ratio, $\eta := \frac{m_1 m_2}{m^2}$, $m = m_1 + m_2$ as the sum of the masses
δ_mthe difference in the masses, $\delta_m := m_1 - m_2$
$ E $absolute value of the scaled orbital binding energy
GNewtons gravitational constant
cspeed of light

Multiple indices are used according to Thorne and Blanchet [17, 23] . In the final form of our results, energies are scaled with μc^2 , separations with $\frac{Gm}{c^2}$, momenta with μc and times with $\frac{Gm}{c^3}$, such that all quantities are dimensionless. Lower index “0” labels the value of a quantity at time $t = t_0$.

Schematic prescription of our calculation

For convenience, we show a flow diagram of our computation. Preliminary results have been surrounded by boxes.



The single steps are being detailed below.

2 The binary orbital motion: 2PN accurate Quasi-Keplerian parameterization

To compute the time derivatives of the multipole moments which we do in harmonic coordinates, we take the orbital parameterization at 2PN from [3],

$$r = a_r (1 - e_r \cos u) , \quad (1)$$

$$\mathcal{M} \equiv \mathcal{N}(t - t_0) = u - e_t \sin u + \epsilon^4 [F_{v-u}(v - u) + F_v \sin v] , \quad (2)$$

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + (\epsilon^4 f_{4\phi}) \sin 2v + (\epsilon^4 g_{4\phi}) \sin 3v , \quad (3)$$

where $v = 2 \arctan \left[\left(\frac{1+e_\phi}{1-e_\phi} \right)^{1/2} \tan \frac{u}{2} \right]$. The explicit 2PN accurate expressions for the orbital elements and functions of the generalized quasi-Keplerian parameterization, in harmonic coordinates, read

$$\begin{aligned}
 a_r &= \frac{1}{(2|E|)} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{4} (-7 + \eta) + \epsilon^4 \frac{(2|E|)^2}{16} \left[1 + \eta^2 + \frac{16}{(2|E|h^2)} (-4 + 7\eta) \right] \right\}, \\
 e_r^2 &= 1 - 2|E|h^2 + \epsilon^2 \frac{(2|E|)}{4} \left\{ 24 - 4\eta + 5(-3 + \eta)(2|E|h^2) \right\}
 \end{aligned} \quad (4)$$

$$+ \epsilon^4 \frac{(2|E|)^2}{8} \left\{ 60 + 148\eta + 2\eta^2 - (2|E|h^2)(80 - 45\eta + 4\eta^2) + \frac{32}{(2|E|h^2)}(4 - 7\eta) \right\}, \quad (5)$$

$$\mathcal{N} = (2|E|)^{3/2} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{8} (-15 + \eta) + \epsilon^4 \frac{(2|E|)^2}{128} \left[555 + 30\eta + 11\eta^2 + \frac{192}{\sqrt{(2|E|h^2)}}(-5 + 2\eta) \right] \right\}, \quad (6)$$

$$\mathcal{K} = \underbrace{\epsilon^2 \frac{6|E|}{\mathcal{W}_{e_t}^2}}_{\mathcal{K}_{1\text{PN}}} + \underbrace{\epsilon^4 \frac{-3|E|^2((9\eta - 22)e_t^2 + 9\eta - 21)}{\mathcal{W}_{e_t}^4}}_{\mathcal{K}_{2\text{PN}}}, \quad (7)$$

$$\begin{aligned} e_t^2 &= 1 - 2|E|h^2 + \epsilon^2 \frac{(2|E|)}{4} \left\{ -8 + 8\eta - (2|E|h^2)(-17 + 7\eta) \right\} \\ &+ \epsilon^4 \frac{(2|E|)^2}{8} \left\{ 12 + 72\eta + 20\eta^2 - 24\sqrt{(2|E|h^2)}(-5 + 2\eta) \right. \\ &- (2|E|h^2)(112 - 47\eta + 16\eta^2) - \frac{16}{(2|E|h^2)}(-4 + 7\eta) \\ &\left. + \frac{24}{\sqrt{(2|E|h^2)}}(-5 + 2\eta) \right\}, \end{aligned} \quad (8)$$

$$F_{v-u} = -\frac{3(2|E|)^2}{2} \left\{ \frac{1}{\sqrt{(2|E|h^2)}}(-5 + 2\eta) \right\}, \quad (9)$$

$$F_v = -\frac{(2|E|)^2}{8} \left\{ \frac{\sqrt{1-2|E|h^2}}{\sqrt{(2|E|h^2)}}\eta(-15 + \eta) \right\}, \quad (10)$$

$$\Phi = 2\pi \left\{ 1 + \epsilon^2 \frac{3}{h^2} + \epsilon^4 \frac{(2|E|)^2}{4} \left[\frac{3}{(2|E|h^2)}(-5 + 2\eta) - \frac{15}{(2|E|h^2)^2}(-7 + 2\eta) \right] \right\}, \quad (11)$$

$$f_{4\phi} = \frac{(2|E|)^2}{8} \left\{ \frac{1-2|E|h^2}{(2|E|h^2)^2} (1 + 19\eta - 3\eta^2) \right\}, \quad (12)$$

$$g_{4\phi} = -\frac{(2|E|)^2}{32} \left\{ \frac{(1-2|E|h^2)^{3/2}}{(2|E|h^2)^2} \eta(-1 + 3\eta) \right\}, \quad (13)$$

$$\begin{aligned} e_\phi^2 &= 1 - 2|E|h^2 + \epsilon^2 \frac{(2|E|)}{4} \left\{ 24 + (2|E|h^2)(-15 + \eta) \right\} \\ &+ \epsilon^4 \frac{(2|E|)^2}{16} \left\{ -40 + 34\eta + 18\eta^2 - (2|E|h^2)(160 - 31\eta + 3\eta^2) \right. \\ &\left. - \frac{1}{(2|E|h^2)}(-416 + 91\eta + 15\eta^2) \right\}, \end{aligned} \quad (14)$$

where we introduced the abbreviation

$$\mathcal{W}_{e_t} := \sqrt{1 - e_t^2}. \quad (15)$$

There are 2PN accurate relations connecting the three eccentricities e_r , e_t and e_ϕ . These relations read

$$\begin{aligned} e_t &= e_r \left\{ 1 + \frac{(2|E|)}{2c^2}(3\eta - 8) + \frac{(2|E|)^2}{4c^4} \frac{1}{(2|E|h^2)} \left[-16 + 28\eta \right. \right. \\ &\left. \left. + (-30 + 12\eta)\sqrt{(2|E|h^2)} + (36 - 19\eta + 6\eta^2)(2|E|h^2) \right] \right\}, \end{aligned} \quad (16a)$$

$$e_\phi = e_r \left\{ 1 + \epsilon^2 \frac{(2|E|)}{2} \eta + \epsilon^4 \frac{(2|E|)^2}{32} \frac{1}{(2|E|h^2)} \left[160 + 357\eta - 15\eta^2 + (-\eta + 11\eta^2)(2|E|h^2) \right] \right\}. \quad (16b)$$

Note that the set $(|E|, e_t, \phi_0)$ is completely sufficient to describe the initial data of the system in its plane perpendicular to \mathbf{h} , thus, we eliminate h for the benefit of e_t in each expression with the help of Equation (8). We could also have worked with the QKP in ADM coordinates, but this would have implied a transformation of the multipole moments. Reference [22] provided this transformation for non-spinning compact binaries, coming from a change in the positions \mathbf{x} and the velocities \mathbf{v} at 2PN. Reference [24] supplied the associated spin-dependent terms to 2PN (as we count the spins of order $\mathcal{O}(\epsilon^0)$). Working in harmonic coordinates makes life more easy as harmonic coordinates are a kind of “natural environment” for the GW prescription and, there, those transformations disappear. In the next section, we list the relevant GW moments and their time derivatives (thanks to the work of Blanchet, Damour and Iyer [23, 25, 21]. Reviews of this subject can be found in [19] and [26]).

3 The relevant multipoles and their time derivatives in harmonic coordinates

The far-zone regime GW forms, regarding only the instantaneous (non-tail) parts, can be expressed in terms of the mass- and current-type multipole moments [17, 21],

$$h_{km}^{\text{TT}}(\mathbf{R}, t) = \frac{G}{c^4 R} \mathcal{P}_{kmij}(\mathbf{N}) \sum_{l=2}^{\infty} \left[\left(\frac{1}{c} \right)^{l-2} \left(\frac{4}{l!} \right) \mathcal{I}_{ij\mathcal{A}_{l-2}}^{(l)}(t - R/c) N_{\mathcal{A}_{l-2}} + \left(\frac{1}{c} \right)^{l-1} \left(\frac{8l}{(l+1)!} \right) \epsilon_{pq(i} \mathcal{J}_{j)p\mathcal{A}_{l-2}}^{(l)}(t - R/c) N_{q\mathcal{A}_{l-2}} \right]. \quad (17)$$

Here, \mathcal{A}_l is a multi-index with $\mathcal{A}_l \equiv a_1 \dots a_l$, indices with round brackets are symmetrized over, viz. for example $\mathcal{A}_{(ij)} \equiv \frac{1}{2} (\mathcal{A}_{ij} + \mathcal{A}_{ji})$, and the following definitions came to use:

$$\mathbf{R} := \mathbf{N} R, \quad (18)$$

$$\mathcal{I}_{\mathcal{A}_{l-2}} := \mathcal{I}_{a_1 \dots a_{l-2}}, \quad (19)$$

$$N_{\mathcal{A}_{l-2}} := N_{a_1} \dots N_{a_{l-2}}, \quad (20)$$

$$P_{ijkl}(\mathbf{N}) := (\delta_{ik} - N_{ik})(\delta_{jl} - N_{jl}) - \frac{1}{2}(\delta_{ij} - N_{ij})(\delta_{kl} - N_{kl}). \quad (21)$$

R denotes the distance from the observer to the binary. The quantity N with multiple indices is a tensor product of components N_i of the normal vector \mathbf{N} (representing the line of sight from the observer to the center of mass of the binary) from N_{a_1} to $N_{a_{l-2}}$, and the superscript (l) denotes the l^{th} time derivative. In the current case of 2PN accurate orbital dynamics, we have to evaluate the GW amplitude, Equation (17), consistently to 2PN order relative to the leading term, which kicks in at ϵ^4 . The mass-type multipoles relevant for the above equation read [22]

$$\begin{aligned} \mathcal{I}_{ij} &= \mu \text{STF}_{ij} \left\{ x^{ij} \left[1 + \right. \right. \\ &+ \frac{1}{42 c^2} \left((29 - 87\eta)v^2 - (30 - 48\eta) \frac{Gm}{r} \right) \\ &+ \left. \left. \frac{1}{c^4} \left(\frac{1}{504} (253 - 1835\eta + 3545\eta^2) v^4 \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{756}(2021 - 5947\eta - 4883\eta^2)\frac{Gm}{r}v^2 \\
& - \frac{1}{756}(131 - 907\eta + 1273\eta^2)\frac{Gm}{r}\dot{r}^2 \\
& - \frac{1}{252}(355 + 1906\eta - 337\eta^2)\frac{G^2m^2}{r^2} \Big] \\
& - x^i v^j \left[\frac{r\dot{r}}{42c^2}(24 - 72\eta) \right. \\
& + \frac{r\dot{r}}{c^4} \left(\frac{1}{63}(26 - 202\eta + 418\eta^2)v^2 \right. \\
& + \left. \frac{1}{378}(1085 - 4057\eta - 1463\eta^2)\frac{Gm}{r} \right) \Big] \\
& + v^{ij} \left[\frac{r^2}{21c^2}(11 - 33\eta) \right. \\
& + \frac{r^2}{c^4} \left(\frac{1}{126}(41 - 337\eta + 733\eta^2)v^2 \right. \\
& + \frac{5}{63}(1 - 5\eta + 5\eta^2)\dot{r}^2 \\
& + \left. \left. \frac{1}{189}(742 - 335\eta - 985\eta^2)\frac{Gm}{r} \right) \right] \Big] , \tag{22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{ijk} = & - \left(\mu \frac{\delta_m}{m} \right) \text{STF}_{ijk} \left\{ \right. \\
& x^{ijk} \left[1 + \frac{1}{6c^2} \left((5 - 19\eta)v^2 - (5 - 13\eta)\frac{Gm}{r} \right) \right] \\
& - x^{ij} v^k \left[\frac{r\dot{r}}{c^2}(1 - 2\eta) \right] \\
& + \left. x^i v^{jk} \left[\frac{r^2}{c^2}(1 - 2\eta) \right] \right\} , \tag{23}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{ijkl} = & \mu \text{STF}_{ijkl} \left\{ \right. \\
& x^{ijkl} \left[(1 - 3\eta) \right. \\
& + \frac{1}{110c^2} \left((103 - 735\eta + 1395\eta^2)v^2 \right. \\
& - \left. \left. (100 - 610\eta + 1050\eta^2)\frac{Gm}{r} \right) \right] \\
& - v^i x^{jkl} \left\{ \frac{72r\dot{r}}{55c^2}(1 - 5\eta + 5\eta^2) \right\} \\
& + \left. v^{ij} x^{kl} \left\{ \frac{78r^2}{55c^2}(1 - 5\eta + 5\eta^2) \right\} \right\} , \tag{24}
\end{aligned}$$

$$\mathcal{I}_{ijklm} = - \left(\mu \frac{\delta_m}{m} \right) (1 - 2\eta) \text{STF}_{ijklm} \{ x^{ijklm} \} , \tag{25}$$

$$\mathcal{I}_{ijklmn} = \mu(1 - 5\eta + 5\eta^2) \text{STF}_{ijklmn} \{ x^{ijklmn} \} . \tag{26}$$

The current-type moments read

$$\begin{aligned} \mathcal{J}_{ij} = & - \left(\mu \frac{\delta_m}{m} \right) \text{STF}_{ij} \epsilon_{jab} \left\{ \right. \\ & x^{ia} v^b \left[1 + \frac{1}{28 c^2} \left((13 - 68\eta) v^2 \right. \right. \\ & \left. \left. + (54 + 60\eta) \frac{Gm}{r} \right) \right] \\ & \left. + v^{ib} x^a \left[\frac{r \dot{r}}{28 c^2} (5 - 10\eta) \right] \right\}, \end{aligned} \quad (27)$$

$$\mathcal{J}_{ijkl} = - \left(\mu \frac{\delta_m}{m} (1 - 2\eta) \right) \text{STF}_{ijkl} \{ \epsilon_{lab} x^{ajk} v^b \}, \quad (28)$$

$$\mathcal{J}_{ijklm} = (\mu (1 - 5\eta + 5\eta^2)) \text{STF}_{ijklm} \{ \epsilon_{mab} x^{ajkl} v^b \}. \quad (29)$$

The notation $\text{STF}_{ij\dots}$ denotes the symmetric trace-free part of the tensor with indices $ij\dots$. The GW amplitude, and from that computed, the far zone angular momentum and energy transport, has been completed to 3PN in [27, 28, 29]. From Thorne's paper [17], see his Equation (4.3), we also extract that the GW amplitude can equivalently be expressed in terms of tensor spherical harmonics,

$$\begin{aligned} h_{jk}^{\text{TT}} = & \frac{1}{c^4 R} \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} \left[\left(\frac{1}{c} \right)^{l-2} I^{lm}(t - R/c) T_{jk}^{E2,lm}(\theta, \phi) \right. \\ & \left. + \left(\frac{1}{c} \right)^{l-1} S^{lm}(t - R/c) T_{jk}^{B2,lm}(\theta, \phi) \right]. \end{aligned} \quad (30)$$

The components I^{lm} and S^{lm} are projected out of Equation (17) with the help of the spherical components $Y_{A_l}^{lm}$. The basis transformation and the explicit representation of the $Y_{A_l}^{lm}$, taking the direction of the orbital angular momentum as the preferred one, $\mathbf{h} = h \mathbf{e}_z$, reads

$$I^{lm}(t) = \frac{16\pi}{(2l+1)!!} \left[\frac{(l+1)(l+2)}{2(l-1)l} \right]^{1/2} \mathcal{I}_{A_l}(t) Y_{A_l}^{lm*}, \quad (31)$$

$$S^{lm}(t) = \frac{-32\pi}{(l+1)(2l+1)!!} \left[\frac{(l+1)(l+2)}{2(l-1)l} \right]^{1/2} \mathcal{J}_{A_l}(t) Y_{A_l}^{lm*}, \quad (32)$$

$$\begin{aligned} Y_{A_l}^{lm} & \stackrel{A}{=} (-1)^m (2l-1)!! \left[\frac{2l+1}{4\pi(l-m)!(l+m)!} \right]^{1/2} \left(\delta_{i_1}^1 + i\delta_{i_1}^2 \right) \dots \left(\delta_{i_m}^1 + i\delta_{i_m}^2 \right) \delta_{i_{m+1}}^3 \dots \delta_{i_l}^3 \\ & \stackrel{B}{=} (-1)^m \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} \sum_{k=0}^{\lfloor \frac{1}{2}(l-m) \rfloor} \frac{(-1)^k (2l-2k-1)!!}{(l-m-2k)!(2k)!!} \times \\ & \quad \delta_{i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} \left(\delta_{i_{2k+1}}^1 + i\delta_{i_{2k+1}}^2 \right) \dots \left(\delta_{i_{2k+m}}^1 + i\delta_{i_{2k+m}}^2 \right) \dots \delta_{i_{2k+m+1}}^3 \dots \delta_{i_l}^3, \end{aligned} \quad (33)$$

$$Y_{A_l}^{lm*} = (-1)^m Y_{A_l}^{l|m|*} \text{ for } (m < 0), \quad (34)$$

where the notation “ $[x]$ ” denotes the integer number which is the largest smaller than or equal to x . From indices comprised by \langle and \rangle , the STF parts have to be taken. The number l tells us what type of moment we have: quadrupole for $l = 2$, octupole for $l = 3$, hexadecapole for $l = 4$, \dots . From Equation (34) we see that we only have to compute the moments for $m = 0 \dots l$. Equal sign B above, taken from Appendix (A3) of [23], may be more practical for programming than A. The relevant pure-spin tensor harmonics, $T_{jk}^{E2,lm}$ and $T_{jk}^{B2,lm}$, are given in our Appendix C. The

reader can find the 1PN outputs in [20], which we also used for our prequel paper. In case of 2PN accurate conservative dynamics, the GW amplitude, consistently worked out to the same relative order in $c^{-1} = \epsilon$ and having moved to our scaling, explicitly reads

$$\begin{aligned}
h_{ij}^{TT} = & \frac{\epsilon^4}{R} \left\{ \sum_{m=-2}^2 I^{(2)2m} T_{ij}^{E2,2m} + \epsilon \left[\sum_{m=-2}^2 S^{(2)2m} T_{ij}^{B2,2m} + \sum_{m=-3}^3 I^{(3)3m} T_{ij}^{E2,3m} \right] \right. \\
& + \epsilon^2 \left[\sum_{m=-3}^3 S^{(3)3m} T_{ij}^{B2,3m} + \sum_{m=-4}^4 I^{(4)4m} T_{ij}^{E2,4m} \right] \\
& + \epsilon^3 \left[\sum_{m=-4}^4 S^{(4)4m} T_{ij}^{B2,4m} + \sum_{m=-5}^5 I^{(5)5m} T_{ij}^{E2,5m} \right] \\
& \left. + \epsilon^4 \left[\sum_{m=-5}^5 S^{(5)5m} T_{ij}^{B2,5m} + \sum_{m=-6}^6 I^{(6)6m} T_{ij}^{E2,6m} \right] \right\}. \tag{35}
\end{aligned}$$

To reach there, we first have to compute the l^{th} time derivatives of the STF mass and current moments. We obtain them by means of the accelerations for a compact binary in harmonic coordinates to 2PN order, $\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{1\text{PN}} + \mathbf{a}_{2\text{PN}}$, (also taken from [22], written in *their units*),

$$\mathbf{a}_N = -\frac{Gm}{r^2} \mathbf{n}_{12}, \tag{36}$$

$$\mathbf{a}_{1\text{PN}} = -\frac{Gm}{c^2 r^2} \left\{ \left[-2(2+\eta) \frac{Gm}{r} + (1+3\eta)v^2 - \frac{3}{2}\eta \dot{r}^2 \right] \mathbf{n}_{12} - 2(2-\eta) \dot{r} \mathbf{v} \right\}, \tag{37}$$

$$\begin{aligned}
\mathbf{a}_{2\text{PN}} = & -\frac{Gm}{c^4 r^2} \left\{ \left[\frac{3}{4}(12+29\eta) \frac{G^2 m^2}{r^2} + \eta(3-4\eta)v^4 + \frac{15}{8}\eta(1-3\eta)\dot{r}^4 \right. \right. \\
& - \frac{3}{2}\eta(3-4\eta)v^2 \dot{r}^2 - \frac{1}{2}\eta(13-4\eta) \frac{Gm}{r} v^2 - (2+25\eta+2\eta^2) \frac{Gm}{r} \dot{r}^2 \left. \right] \mathbf{n}_{12} \\
& \left. - \frac{1}{2} \left[\eta(15+4\eta)v^2 - (4+41\eta+8\eta^2) \frac{Gm}{r} - 3\eta(3+2\eta)\dot{r}^2 \right] \dot{r} \mathbf{v} \right\}. \tag{38}
\end{aligned}$$

The results are lengthy, easily reproducible and needed only temporarily; thus, they are not listed here. We employ the orbital parameterization, Equations (1) – (16b), to obtain the normal and the velocity vector \mathbf{n}_{12} and \mathbf{v} in spherical coordinates, symbolically

$$\mathbf{r} = \mathbf{n}_{12} r, \tag{39}$$

$$\mathbf{n}_{12} = \{\cos(\phi), \sin(\phi), 0\}, \tag{40}$$

$$\mathbf{v} := \frac{d\mathbf{r}}{dt} = \left[\frac{\partial \mathbf{r}}{\partial u} \right] \left[\frac{\partial u}{\partial \mathcal{M}} \right] \left[\frac{\partial \mathcal{M}}{\partial t} \right]. \tag{41}$$

Equation (41) is to be computed with the help of the KE (2). Again, it is not necessary to provide the velocity as functions of u and ϕ because these terms are easy to be reproduced. With this input we can compute the spherical tensor components of Equation (35). The next section gives the results, using the 2PN accurate QKP in harmonic coordinates.

4 The GW forms: tensor spherical components of h_{ij}^{TT}

Defining

$$A(u) := 1 - e_t \cos u, \tag{42}$$

we get

$$\begin{aligned}
I^{(2)20} = & 16\sqrt{\frac{\pi}{15}}|E|\mu\left\{1 - \frac{1}{A(u)} + |E|\epsilon^2\left[-\frac{2(\eta-26)\mathcal{W}_{e_t}^2}{7A(u)^3} + \frac{3(3\eta-1)}{14A(u)} + \frac{2(\eta-26)}{7A(u)^2}\right.\right. \\
& - \frac{3}{14}(3\eta-1)\left.\right] + \epsilon^4|E|^2\left[\frac{1}{42}(5-\eta(11\eta+16)) + \frac{1}{42\mathcal{W}_{e_t}^2A(u)}\left(252\mathcal{W}_{e_t}(5-2\eta)\right.\right. \\
& - (\eta(11\eta+16)-5)(e_t^2-1)\left.\right) + \frac{1}{126\mathcal{W}_{e_t}^2A(u)^2}\left((1877-323\eta)\eta-3682\right)e_t^2 \\
& + 756\mathcal{W}_{e_t}(2\eta-5) + 323\eta^2 + 1651\eta + 1666\left.\right) + \frac{1}{21A(u)^3}\left((686-2\eta(31\eta+197))e_t^2\right. \\
& + \eta(41\eta+625)-1610\left.\right) - \frac{\mathcal{W}_{e_t}^2(\eta(1517\eta+7549)-8645)}{126A(u)^4} \\
& \left. + \frac{\mathcal{W}_{e_t}^4(4\eta(79\eta+179)-217)}{42A(u)^5}\right\}, \tag{43}
\end{aligned}$$

$$I^{(2)21} = 0, \tag{44}$$

$$\begin{aligned}
I^{(2)22} = & 8\sqrt{\frac{2\pi}{5}}|E|\mu e^{-2i\phi}\left\{\frac{1}{A(u)} - 1 - \frac{2\mathcal{W}_{e_t}^2 + 2ie_t\sin(u)\mathcal{W}_{e_t}}{A(u)^2} + \epsilon^2|E|\left[\frac{3(1-3\eta)}{14A(u)} + \frac{3}{14}(3\eta-1)\right.\right. \\
& + \frac{\frac{2}{21}(15\eta-82)\mathcal{W}_{e_t}^2 + \frac{2}{21}ie_t(6\eta-79)\sin(u)\mathcal{W}_{e_t}}{A(u)^3} + A(u)^{-2}\left\{\frac{1}{14(e_t-1)}(98\eta-210\right. \\
& + e_t(35(5-3\eta) + 3e_t(5e_t+2)(3\eta-1))) - \frac{1}{14\mathcal{W}_{e_t}}e_t\left(7\sqrt{\frac{1+e_t}{1-e_t}}((3-9\eta)e_t^2 + \eta+5)\right. \\
& \left.\left.+ 2i((9\eta-3)e_t^2 + 19\eta-25)\sin(u)\right)\right\}\left.\right] \\
& + \epsilon^4|E|^2\left[ie_t\sin(u)\left(\frac{\mathcal{W}_{e_t}^3(614\eta^2-3710\eta+427)}{63A(u)^5} - \frac{\mathcal{W}_{e_t}(\eta(4049\eta+3901)-10016)}{189A(u)^4}\right.\right. \\
& - \frac{((\eta(377\eta-575)+2200)e_t^2 - \eta(3024\mathcal{W}_{e_t}+449\eta+1933)+7560\mathcal{W}_{e_t}-1132)}{63\mathcal{W}_{e_t}A(u)^3} \\
& + \frac{1}{21\mathcal{W}_{e_t}^3A(u)^2}\left((\eta(11\eta+16)-5)e_t^4 + 2(\eta(67\eta+398)+59)e_t^2 + 252\mathcal{W}_{e_t}(5-2\eta)\right. \\
& - 61\eta^2 + 196\eta - 701)\left.\right) + \frac{1}{42}(\eta(11\eta+16)-5) + \frac{\frac{6(2\eta-5)}{\mathcal{W}_{e_t}} + \frac{1}{42}(5-\eta(11\eta+16))}{A(u)} \\
& + \frac{1}{126\mathcal{W}_{e_t}^2A(u)^2}\left(6(\eta(11\eta+16)-5)e_t^4 + (\eta(2063\eta+7867)+5038)e_t^2\right. \\
& - 3780\mathcal{W}_{e_t}(2\eta-5) - 2129\eta^2 + 2621\eta - 11056\left.\right) \\
& + \frac{(4(169-64\eta)\eta-766)e_t^2 + 504\mathcal{W}_{e_t}(2\eta-5) + 373\eta^2 + 1317\eta + 378}{21A(u)^3} \\
& \left. + \frac{\mathcal{W}_{e_t}^2((1247-12113\eta)\eta+28061)}{378A(u)^4} + \frac{\mathcal{W}_{e_t}^4(16\eta(143\eta-335)+199)}{126A(u)^5}\right]\left.\right\}, \tag{45}
\end{aligned}$$

$$\overset{(2)}{S}^{20} = 0, \quad (46)$$

$$\begin{aligned} \overset{(2)}{S}^{21} = & \frac{32}{3} \sqrt{\frac{\pi}{5}} |E|^{3/2} \mathcal{W}_{e_t} \eta \delta_m e^{-i\phi} \left\{ \frac{1}{A(u)^2} \right. \\ & + |E| \epsilon^2 \left[\frac{((675 - 3639\eta)e_t^2 + 3695\eta - 731)}{28 \mathcal{W}_{e_t}^2 A(u)^2} + \frac{4(\eta - 1)}{A(u)^3} \right. \\ & \left. \left. + A(u)^{-4} \left\{ \frac{1}{14} \mathcal{W}_{e_t}^2 (2245 - 2096\eta) + \frac{1}{7} i \mathcal{W}_{e_t} (2054\eta - 2077) e_t \sin(u) \right\} \right] \right\}, \end{aligned} \quad (47)$$

$$\overset{(2)}{S}^{22} = 0, \quad (48)$$

$$\overset{(3)}{I}^{30} = 0, \quad (49)$$

$$\begin{aligned} \overset{(3)}{I}^{31} = & \frac{4i \sqrt{\frac{10\pi}{7}} |E|^{3/2} \mu \mathcal{W}_{e_t} \delta_m}{3m} e^{-i\phi} \left\{ \frac{-6 - \frac{6ie_t \sin(u)}{\mathcal{W}_{e_t}}}{5A(u)} + \frac{1}{A(u)^2} \right. \\ & + |E| \epsilon^2 \left[-\frac{((5\eta - 7)e_t^2 + 19\eta - 17) - i \mathcal{W}_{e_t} (5\eta - 7) e_t \sin(u)}{10 \mathcal{W}_{e_t}^2 A(u)} \right. \\ & - \frac{(5\eta - 79)e_t^2 - 45\eta + 119}{20 \mathcal{W}_{e_t}^2 A(u)^2} + \frac{-4 \mathcal{W}_{e_t} (\eta - 14) - 2i(4\eta - 17) e_t \sin(u)}{3 \mathcal{W}_{e_t} A(u)^3} \\ & \left. \left. + \frac{3 \mathcal{W}_{e_t}^2 (2\eta - 23) - 2i \mathcal{W}_{e_t} (\eta + 10) e_t \sin(u)}{5A(u)^4} \right] \right\}, \end{aligned} \quad (50)$$

$$\overset{(3)}{I}^{32} = 0, \quad (51)$$

$$\begin{aligned} \overset{(3)}{I}^{33} = & -20i \sqrt{\frac{2\pi}{21}} |E|^{3/2} \mathcal{W}_{e_t} \eta \delta_m e^{-3i\phi} \left\{ \frac{1}{A(u)^2} - \frac{8 \mathcal{W}_{e_t} (\mathcal{W}_{e_t} + ie_t \sin(u))}{5A(u)^3} \right. \\ & + \frac{-\frac{6}{5} - \frac{2ie_t \sin(u)}{5 \mathcal{W}_{e_t}}}{A(u)} + |E| \epsilon^2 \left[\frac{i \mathcal{W}_{e_t} (5\eta - 7) e_t \sin(u) - 3(5\eta - 7) e_t^2 - 57\eta + 51}{30 \mathcal{W}_{e_t}^2 A(u)} \right. \\ & + \frac{(79 - 5\eta) e_t^2 + 45\eta - 119}{20 \mathcal{W}_{e_t}^2 A(u)^2} - \{45 \mathcal{W}_{e_t} A(u)^3\}^{-1} \left\{ 6 \mathcal{W}_{e_t} ((5\eta - 7) e_t^2 + 77\eta - 205) \right. \\ & + 2ie_t \sin(u) (3(5\eta - 7) e_t^2 + 149\eta - 208) \left. \right\} \\ & \left. \left. + \frac{\mathcal{W}_{e_t} (\mathcal{W}_{e_t} (34\eta - 191) + 14i(\eta - 14) e_t \sin(u))}{15A(u)^4} \right] \right\}, \end{aligned} \quad (52)$$

$$\begin{aligned} \overset{(3)}{S}^{30} = & 16 \sqrt{\frac{\pi}{105}} |E|^2 m \mathcal{W}_{e_t} \eta (3\eta - 1) e_t \left\{ \frac{\sin(u)}{A(u)^3} \right. \\ & + |E| \epsilon^2 \left[\frac{\sin(u) ((8\eta^2 - 17\eta + 4) e_t^2 + 10\eta^2 - 7\eta + 2)}{3 \mathcal{W}_{e_t}^2 (3\eta - 1) A(u)^3} \right. \\ & \left. \left. + \frac{2(2\eta^2 - 242\eta + 79) \sin(u)}{(9 - 27\eta) A(u)^4} - \frac{\mathcal{W}_{e_t}^2 (2\eta^2 - 242\eta + 79) \sin(u)}{(3 - 9\eta) A(u)^5} \right] \right\}, \end{aligned} \quad (53)$$

$$S^{(3)31} = 0, \quad (54)$$

$$S^{(3)32} = \frac{32}{3} \sqrt{\frac{2\pi}{7}} |E|^2 \mu \mathcal{W}_{e_t}^2 (3\eta - 1) e^{-2i\phi} \left\{ \frac{i}{A(u)^3} - \frac{e_t \sin(u)}{4 \mathcal{W}_{e_t} A(u)^3} \right. \\ \left. + |E| \epsilon^2 \left[\sin(u) \left(-\frac{e_t ((8\eta^2 - 17\eta + 4) e_t^2 + 10\eta^2 - 7\eta + 2)}{12 \mathcal{W}_{e_t}^3 (3\eta - 1) A(u)^3} \right. \right. \right. \\ \left. + \frac{(2\eta^2 - 242\eta + 79) e_t}{18 \mathcal{W}_{e_t} (3\eta - 1) A(u)^4} + \frac{\mathcal{W}_{e_t} (22\eta^2 + 338\eta - 109) e_t}{(12 - 36\eta) A(u)^5} \right) \\ \left. + \frac{i (15(1 - 3\eta)^2 e_t^2 + 5 (19 \mathcal{W}_{e_t}^2 - 3) \eta^2 - 10 (2 \mathcal{W}_{e_t}^2 + 7) \eta - \mathcal{W}_{e_t}^2 + 25)}{10 \mathcal{W}_{e_t}^2 (3\eta - 1) A(u)^3} \right. \\ \left. + \frac{i (197\eta^2 - 239\eta + 55)}{(54\eta - 18) A(u)^4} + \frac{i \mathcal{W}_{e_t}^2 (59\eta^2 - 89\eta + 25)}{(6 - 18\eta) A(u)^5} \right] \left. \right\}, \quad (55)$$

$$S^{(3)33} = 0, \quad (56)$$

$$I^{(4)40} = \frac{8}{21} \sqrt{\frac{\pi}{5}} |E|^2 \mu (1 - 3\eta) \left\{ \frac{5 - 5e_t^2}{A(u)^3} - \frac{6}{A(u)} - \frac{5}{A(u)^2} + 6 \right. \\ \left. + |E| \epsilon^2 \left[\frac{9 \mathcal{W}_{e_t}^4 (20\eta(7\eta - 128) + 831)}{110(3\eta - 1) A(u)^5} - \frac{3 \mathcal{W}_{e_t}^2 (20\eta(7\eta - 128) + 831)}{(66\eta - 22) A(u)^4} \right. \right. \\ \left. + \frac{\mathcal{W}_{e_t}^2 (5\eta(2\eta + 163) - 218)}{55(3\eta - 1) A(u)^3} + \frac{5\eta(82\eta - 1699) + 2711}{55(3\eta - 1) A(u)^2} \right. \\ \left. + \frac{6(5\eta(8\eta - 41) + 52)}{55(3\eta - 1) A(u)} - \frac{6(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)} \right] \left. \right\}, \quad (57)$$

$$I^{(4)41} = 0, \quad (58)$$

$$I^{(4)42} = \frac{16}{21} \sqrt{2\pi} |E|^2 m \eta (3\eta - 1) e^{-2i\phi} \left\{ 1 - \frac{1}{A(u)} + \frac{-\mathcal{W}_{e_t}^2 + i \mathcal{W}_{e_t} e_t \sin(u)}{2A(u)^3} \right. \\ \left. + \frac{2i \mathcal{W}_{e_t} e_t \sin(u) - 2e_t^2 + \frac{7}{6}}{A(u)^2} \right. \\ \left. + |E| \epsilon^2 \left[\{88(3\eta - 1) A(u)^5\}^{-1} \mathcal{W}_{e_t} \left\{ -22 \mathcal{W}_{e_t}^4 (30\eta^2 - 250\eta + 79) \right. \right. \right. \\ \left. - 5i \mathcal{W}_{e_t}^2 (7\eta^2 - 447\eta + 151) e_t \sin(u) \right\} \\ \left. + \frac{\mathcal{W}_{e_t}^2 (928\eta^2 - 11620\eta + 3743) + i \mathcal{W}_{e_t} (2147\eta^2 - 10331\eta + 3175) e_t \sin(u)}{132(3\eta - 1) A(u)^4} \right. \\ \left. + (330 \mathcal{W}_{e_t}^2 (3\eta - 1) A(u)^2)^{-1} \left(e_t^2 (-75 (53 \mathcal{W}_{e_t}^2 - 66) \eta^2 \right. \right. \\ \left. + 30 (17 \mathcal{W}_{e_t}^2 - 22) \eta + 129 \mathcal{W}_{e_t}^2 - 330) \right. \\ \left. + 12i \mathcal{W}_{e_t} e_t \sin(u) ((40\eta^2 - 205\eta + 52) e_t^2 + 290\eta^2 - 235\eta + 58) \right. \\ \left. - 495(1 - 3\eta)^2 e_t^4 + 8345 \mathcal{W}_{e_t}^2 \eta^2 \right. \end{math>$$

$$\begin{aligned}
& -14285 \mathcal{W}_{e_t}^2 \eta + 3902 \mathcal{W}_{e_t}^2 - 495 \eta^2 - 2310 \eta + 825) \\
& + (660 \mathcal{W}_{e_t}^2 (3\eta - 1) A(u)^3)^{-1} \left(22(e_t^2 (15 (29 \mathcal{W}_{e_t}^2 - 20) \eta^2 \right. \\
& - 20 (75 \mathcal{W}_{e_t}^2 - 2) \eta + 459 \mathcal{W}_{e_t}^2 + 20) \\
& + 30(1 - 3\eta)^2 e_t^4 + (30 - 675 \mathcal{W}_{e_t}^2) \eta^2 + 140 (13 \mathcal{W}_{e_t}^2 + 1) \eta - 539 \mathcal{W}_{e_t}^2 - 50) \\
& + 3i \mathcal{W}_{e_t} e_t \sin(u) ((985 \eta^2 - 5715 \eta + 1781) e_t^2 - 325 \eta^2 + 4835 \eta - 1561) \Big) \\
& \left. + \frac{40 \eta^2 - 205 \eta + 52}{55(3\eta - 1)A(u)} + \frac{40 \eta^2 - 205 \eta + 52}{55 - 165 \eta} \right] \Big\}, \tag{59}
\end{aligned}$$

$$I^{(4)}_{43} = 0, \tag{60}$$

$$\begin{aligned}
I^{(4)}_{44} = & \frac{8}{3} \sqrt{\frac{2\pi}{7}} |E|^2 m(1 - 3\eta) \eta e^{-4i\phi} \left\{ 1 - \frac{1}{A(u)} - \frac{8 \mathcal{W}_{e_t}^2 (-\mathcal{W}_{e_t}^2 - i \mathcal{W}_{e_t} e_t \sin(u))}{A(u)^4} \right. \\
& + \frac{2i \mathcal{W}_{e_t} e_t \sin(u) + 9e_t^2 - 9}{2A(u)^3} + \frac{4i \mathcal{W}_{e_t} e_t \sin(u) - 8e_t^2 + \frac{43}{6}}{A(u)^2} \\
& + \epsilon^2 |E| \left[\sin(u) \left(\frac{i \mathcal{W}_{e_t}^3 (923 \eta^2 - 13683 \eta + 4523) e_t}{44A(u)^5} \right. \right. \\
& - \frac{i \mathcal{W}_{e_t} e_t (48 (40 \eta^2 - 205 \eta + 52) e_t^2 + 56335 \eta^2 - 105175 \eta + 29219)}{330A(u)^4} \\
& - \frac{i e_t ((985 \eta^2 - 5715 \eta + 1781) e_t^2 - 325 \eta^2 + 4835 \eta - 1561)}{110 \mathcal{W}_{e_t} A(u)^3} \\
& \left. \left. - \frac{4i e_t ((40 \eta^2 - 205 \eta + 52) e_t^2 + 290 \eta^2 - 235 \eta + 58)}{55 \mathcal{W}_{e_t} A(u)^2} \right) \right. \\
& + \frac{3 \mathcal{W}_{e_t}^4 (716 \eta^2 - 4456 \eta + 1461)}{44A(u)^5} \\
& - \frac{\mathcal{W}_{e_t}^2 (96 (40 \eta^2 - 205 \eta + 52) e_t^2 + 147740 \eta^2 - 496880 \eta + 149503)}{660A(u)^4} \\
& + (330 \mathcal{W}_{e_t} A(u)^3 (e_t - 1))^{-1} \left\{ -3e_t^3 \left(90 \eta^2 \left(\frac{22(e_t + 1)}{\mathcal{W}_{e_t}} + 5 \mathcal{W}_{e_t} \right) \right. \right. \\
& - 5 \eta \left(\frac{6072(e_t + 1)}{\mathcal{W}_{e_t}} - 2363 \mathcal{W}_{e_t} \right) + \frac{9900(e_t + 1)}{\mathcal{W}_{e_t}} - 3826 \mathcal{W}_{e_t} \Big) \\
& + 1980 (3 \eta^2 - 46 \eta + 15) \sqrt{\frac{1 + e_t}{1 - e_t}} e_t^4 + 3 \mathcal{W}_{e_t} (2430 \eta^2 - 18545 \eta + 6074) e_t^2 \\
& \left. + \mathcal{W}_{e_t} (28410 \eta^2 - 83795 \eta + 25262) e_t + \mathcal{W}_{e_t} (-28410 \eta^2 + 83795 \eta - 25262) \right\}_3 \\
& + (330 \mathcal{W}_{e_t} A(u)^2 (e_t - 1))^{-1} \left\{ -6e_t^3 \left(5 \eta^2 \left(\frac{891(e_t + 1)}{\mathcal{W}_{e_t}} - 827 \mathcal{W}_{e_t} \right) \right. \right. \\
& \left. \left. + \eta \left(1330 \mathcal{W}_{e_t} - \frac{2970(e_t + 1)}{\mathcal{W}_{e_t}} \right) + \frac{495(e_t + 1)}{\mathcal{W}_{e_t}} - 79 \mathcal{W}_{e_t} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& -6e_t^2 \left(5\eta^2 \left(\frac{99(e_t+1)}{\mathcal{W}_{e_t}} - 64\mathcal{W}_{e_t} \right) + 10\eta \left(\frac{231(e_t+1)}{\mathcal{W}_{e_t}} + 164\mathcal{W}_{e_t} \right) \right. \\
& - \frac{825(e_t+1)}{\mathcal{W}_{e_t}} - 416\mathcal{W}_{e_t} \left. \right) + 5e_t \left(\frac{(e_t+1)}{\mathcal{W}_{e_t}} (594\eta^2 + 2772\eta - 990) \right. \\
& + \mathcal{W}_{e_t} (-6628\eta^2 + 5407\eta - 1165) \left. \right) \\
& + 2970(1-3\eta)^2 \sqrt{\frac{1+e_t}{1-e_t}} e_t^4 + 5\mathcal{W}_{e_t} (6034\eta^2 - 8179\eta + 2155) \left. \right\}_2 \\
& + \frac{-40\eta^2 + 205\eta - 52}{55A(u)} + \frac{1}{55} (40\eta^2 - 205\eta + 52) \left. \right\}_0, \tag{61}
\end{aligned}$$

$$S^{(4)40} = 0, \tag{62}$$

$$\begin{aligned}
S^{(4)41} &= \frac{8}{7} \sqrt{2\pi} |E|^{5/2} (1-2\eta) \eta \delta_m e^{-i\phi} \times \\
& \left\{ -\frac{\mathcal{W}_{e_t}^3}{3A(u)^4} + \frac{2\mathcal{W}_{e_t}}{5A(u)^2} - \frac{2ie_t(e_t^2-1)\sin(u)}{3A(u)^4} \right\}, \tag{63}
\end{aligned}$$

$$S^{(4)42} = 0, \tag{64}$$

$$\begin{aligned}
S^{(4)43} &= 40 \sqrt{\frac{2\pi}{7}} |E|^{5/2} (1-2\eta) \eta \delta_m e^{-3i\phi} \times \\
& \left\{ -\frac{\mathcal{W}_{e_t}^3}{3A(u)^4} - \frac{2\mathcal{W}_{e_t}}{75A(u)^2} + \frac{2ie_t(e_t^2-1)\sin(u)}{15A(u)^4} \right\}, \tag{65}
\end{aligned}$$

$$S^{(4)44} = 0, \tag{66}$$

$$I^{(5)50} = 0, \tag{67}$$

$$\begin{aligned}
I^{(5)51} &= \frac{2}{3} \sqrt{\frac{7\pi}{11}} \delta_m |E|^{5/2} \eta (2\eta-1) \left\{ \frac{i\mathcal{W}_{e_t}^3}{A(u)^4} + \frac{8i\mathcal{W}_{e_t}}{7A(u)} - \frac{4i\mathcal{W}_{e_t}}{5A(u)^2} \right. \\
& - \frac{4i\mathcal{W}_{e_t}}{3A(u)^3} + \sin(u) \left(\frac{2e_t}{3A(u)^3} - \frac{8e_t}{7A(u)} \right) \left. \right\}, \tag{68}
\end{aligned}$$

$$I^{(5)52} = 0, \tag{69}$$

$$\begin{aligned}
I^{(5)53} &= \sqrt{\frac{2\pi}{33}} |E|^{5/2} \eta (2\eta-1) \delta_m e^{-3i\phi} \times \\
& \left\{ + \frac{\frac{8}{3}e_t \sin(u) - 8i\mathcal{W}_{e_t}}{A(u)} + \frac{28i\mathcal{W}_{e_t}}{5A(u)^2} - \frac{(e_t^2-1)(16e_t \sin(u) - 5i\mathcal{W}_{e_t})}{3A(u)^4} \right. \\
& + \frac{2(e_t(41-48e_t^2)\sin(u) - 6i(e_t-1)\sqrt{\frac{e_t+1}{1-e_t}}(8e_t^2-1))}{9A(u)^3} \left. \right\}, \tag{70}
\end{aligned}$$

$$I^{(5)54} = 0, \tag{71}$$

$$I^{(5)55} = 8 \sqrt{\frac{2\pi}{165}} |E|^{5/2} \eta (2\eta-1) \delta_m e^{-5i\phi} \times$$

$$\left\{ + \frac{16i \mathcal{W}_{e_t}^4 (\mathcal{W}_{e_t} + ie_t \sin(u))}{A(u)^5} + \frac{3(e_t^2 - 1)(16e_t \sin(u) + 15i \mathcal{W}_{e_t})}{8A(u)^4} \right. \\ \left. + \frac{i(-10 \mathcal{W}_{e_t}(24e_t^2 - 17) + ie_t(137 - 144e_t^2) \sin(u))}{12A(u)^3} \right. \\ \left. - \frac{7i \mathcal{W}_{e_t}}{2A(u)^2} + \frac{-e_t \sin(u) + 5i \mathcal{W}_{e_t}}{A(u)} \right\}, \quad (72)$$

$$\begin{aligned} S^{(5)50} &= \frac{16}{3} \sqrt{\frac{5\pi}{231}} |E|^3 \eta(5(\eta - 1)\eta + 1) m e_t \times \\ &\quad \left\{ \frac{\mathcal{W}_{e_t}^3 \sin(u)}{A(u)^5} - \frac{2 \mathcal{W}_{e_t} \sin(u)}{5A(u)^3} - \frac{2 \mathcal{W}_{e_t} \sin(u)}{3A(u)^4} \right\}, \end{aligned} \quad (73)$$

$$S^{(5)51} = 0, \quad (74)$$

$$\begin{aligned} S^{(5)52} &= \frac{16}{9} \sqrt{\frac{2\pi}{11}} |E|^3 (5(\eta - 1)\eta + 1) \eta m e^{-2i\phi} \times \\ &\quad \left\{ \frac{i \mathcal{W}_{e_t}^4}{A(u)^5} - \frac{i \mathcal{W}_{e_t}^2}{3A(u)^4} - \frac{6i \mathcal{W}_{e_t}^2}{5A(u)^3} + \sin(u) e_t \mathcal{W}_{e_t} \left(\frac{\mathcal{W}_{e_t}^2}{A(u)^5} + \frac{2}{5A(u)^3} + \frac{2}{3A(u)^4} \right) \right\}, \end{aligned} \quad (75)$$

$$S^{(5)53} = 0, \quad (76)$$

$$\begin{aligned} S^{(5)54} &= \frac{16}{15} \sqrt{\frac{2\pi}{33}} |E|^3 \eta(5(\eta - 1)\eta + 1) m e^{-4i\phi} \times \\ &\quad \left\{ \frac{35i \mathcal{W}_{e_t}^4}{A(u)^5} - \frac{6i(e_t^2 - 1)}{A(u)^3} - \frac{5i(e_t^2 - 1)}{3A(u)^4} \right. \\ &\quad \left. + \sin(u) e_t \mathcal{W}_{e_t} \left(-\frac{35 \mathcal{W}_{e_t}^2}{2A(u)^5} - \frac{1}{A(u)^3} - \frac{5}{3A(u)^4} \right) \right\}, \end{aligned} \quad (77)$$

$$S^{(5)55} = 0, \quad (78)$$

$$\begin{aligned} I^{(6)60} &= \frac{64}{33} \sqrt{\frac{10\pi}{273}} |E|^3 \eta(5(\eta - 1)\eta + 1) m \times \\ &\quad \left\{ 1 - \frac{1}{A(u)} - \frac{77}{60A(u)^2} + \frac{7 \mathcal{W}_{e_t}^2}{10A(u)^3} + \frac{35 \mathcal{W}_{e_t}^2}{24A(u)^4} - \frac{7 \mathcal{W}_{e_t}^4}{8A(u)^5} \right\}, \end{aligned} \quad (79)$$

$$I^{(6)61} = 0, \quad (80)$$

$$\begin{aligned} I^{(6)62} &= -\frac{32}{33} \sqrt{\frac{2\pi}{13}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-2i\phi} \times \\ &\quad \left\{ 1 - \frac{1}{A(u)} \frac{43}{60} - \frac{2e_t^2}{A(u)^2} + \frac{23 \mathcal{W}_{e_t}^4}{24A(u)^5} - \frac{\mathcal{W}_{e_t}^2}{10A(u)^3} - \frac{107 \mathcal{W}_{e_t}^2}{72A(u)^4} \right. \\ &\quad \left. + i e_t \mathcal{W}_{e_t} \sin(u) \left(-\frac{\mathcal{W}_{e_t}^2}{12A(u)^5} + \frac{2}{A(u)^2} + \frac{2}{3A(u)^3} - \frac{11}{9A(u)^4} \right) \right\}, \end{aligned} \quad (81)$$

$$I^{(6)63} = 0, \quad (82)$$

$$\begin{aligned}
I_{64}^{(6)} &= \frac{64}{11} \sqrt{\frac{\pi}{195}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-4i\phi} \times \\
&\quad \left\{ 1 - \frac{1}{A(u)} + \frac{\frac{403}{60} - 8e_t^2}{A(u)^2} + \frac{(e_t - 1)(e_t + 1)(576e_t^2 + 167)}{72A(u)^4} \right. \\
&\quad + \frac{91\mathcal{W}_{e_t}^4}{24A(u)^5} - \frac{5\mathcal{W}_{e_t}^2}{2A(u)^3} + \sin(u) \left(\frac{4i\mathcal{W}_{e_t}e_t}{A(u)^2} + \frac{4i\mathcal{W}_{e_t}e_t}{3A(u)^3} \right. \\
&\quad \left. \left. - \frac{2i\mathcal{W}_{e_t}e_t(6e_t - 5)(6e_t + 5)}{9A(u)^4} + \frac{35i\mathcal{W}_{e_t}^3e_t}{6A(u)^5} \right) \right\}, \tag{83}
\end{aligned}$$

$$I_{65}^{(6)} = 0, \tag{84}$$

$$\begin{aligned}
I_{66}^{(6)} &= -\frac{32}{3} \sqrt{\frac{2\pi}{715}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-6i\phi} \times \\
&\quad \left\{ 1 - \frac{1}{A(u)} + \frac{\frac{1003}{60} - 18e_t^2}{A(u)^2} + \frac{32\mathcal{W}_{e_t}^6}{A(u)^6} - \frac{3\mathcal{W}_{e_t}^4}{8A(u)^5} - \frac{13\mathcal{W}_{e_t}^2}{2A(u)^3} - \frac{\mathcal{W}_{e_t}^2(1152e_t^2 - 551)}{24A(u)^4} \right. \\
&\quad \left. + ie_t \sin(u) \left(\frac{32\mathcal{W}_{e_t}^5}{A(u)^6} + \frac{95\mathcal{W}_{e_t}^3}{4A(u)^5} - \frac{\mathcal{W}_{e_t}(96e_t^2 - 85)}{3A(u)^4} + \frac{2\mathcal{W}_{e_t}}{A(u)^3} + \frac{6\mathcal{W}_{e_t}}{A(u)^2} \right) \right\}. \tag{85}
\end{aligned}$$

It will be necessary to decompose these tensor components in terms of irreducible expressions to get a time Fourier representation. Those will be terms which collect contributions having $\sin u$ on the one hand and those without $\sin u$ on the other, and they will be used when we write down the exponential of the orbital phase in such a way that we can use results and representations we already know from the literature or we have to evaluate them from scratch. This computation is subject of the next section.

5 Relevant Kapteyn Series of irreducible components

5.1 Series representation for the inverse KE, $\sin mu$ and $\cos mu$

We recall the computation of the 1PN version of this consideration in paper 1. There we required only a Newtonian accurate expression of the sin- and cos-function of multiples of the eccentric anomaly. For further considerations, let us call $u = g(\mathcal{M})$ the solution to the 2PN KE. As we Taylor expand the argument of the Bessel integral – which will be done below – we are in the position to provide u as a series in \mathcal{M} up to 2PN. Therefore we need the well-known representation of v and u in the KE (2), where in the 2PN term, we can insert their Newtonian accurate summation surrogates (see Equations (5) on p. 553 and (8) on p. 555 in [30]. In [31] there is a misprint in the definition of the G_n on page 33: the factor $\frac{2}{n}$ should comprise the complete right hand side),

$$v_{(e=e_t)} = \mathcal{M} + \sum_{m=1}^{\infty} G_m(e_t) \sin m\mathcal{M}, \tag{86}$$

$$G_m(e) = \frac{2}{m} \left\{ J_m(me) + \sum_{s=1}^{\infty} \alpha^s [J_{m-s}(me) - J_{m+s}(me)] \right\}, \tag{87}$$

and α is extractable from

$$e = \frac{2\alpha}{1 + \alpha^2}. \quad (88)$$

The series expansions of the functions $(v - u)$ and $\sin v$ at Newtonian accuracy read

$$(v - u) = \left(\mathcal{M} + \sum_{i=1}^{\infty} G_i(e_t) \sin(i \mathcal{M}) \right) - \left(\mathcal{M} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n e_t) \sin(n \mathcal{M}) \right) \quad (89)$$

$$= \sum_{i=1}^{\infty} \left(G_i(e_t) - \frac{2}{i} J_i(i e_t) \right) \sin(i \mathcal{M}), \quad (90)$$

$$\sin v = \sqrt{1 - e_t^2} \sum_{n=1}^{\infty} 2 J'_n(n e_t) \sin(n \mathcal{M}). \quad (91)$$

We take above definitions and write in shorthand notation for further calculations, cf. Equation (2),

$$\mathcal{M} = u - e_t \sin u + \epsilon^4 \sum_{j=1}^{\infty} \alpha_j \sin(j \mathcal{M}). \quad (92)$$

Inserting this in the KE and solving for the Fourier-Bessel coefficients, we calculate after Taylor expansion in ϵ (see Appendix A),

$$\begin{aligned} g(\mathcal{M}) - \mathcal{M} &= \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n \mathcal{M}), \\ \mathcal{A}_n &= \frac{2}{n \pi} \int_0^{\pi} \cos(n \mathcal{M}) dg(\mathcal{M}) \\ &= \frac{2}{n \pi} \int_0^{\pi} \cos \left(n \left[g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M})) + \epsilon^4 \sum_{m=1}^{\infty} \alpha_m \sin(\mathcal{M} m) \right] \right) dg(\mathcal{M}) \\ &= \frac{2}{n \pi} \int_0^{\pi} \cos(n [g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M}))]) dg(\mathcal{M}) \\ &\quad - \frac{2}{\pi} \int_0^{\pi} \sin(n [g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M}))]) \epsilon^4 \sum_{m=1}^{\infty} \alpha_m \sin(\mathcal{M} m) dg(\mathcal{M}). \end{aligned} \quad (93)$$

Defining

$$\Theta(j, n) := \begin{cases} 0, & j \leq n \\ 1, & j > n \end{cases}, \quad (95)$$

the result reads

$$\begin{aligned} u &= \mathcal{M} + 2 \left[\sum_{j=1}^{\infty} \frac{\sin(j \mathcal{M}) J_j(j e_t)}{j} \right] \\ &\quad - 2 \epsilon^4 \mathcal{F}_v \sqrt{1 - e_t^2} \sum_{j=1}^{\infty} \sin(j \mathcal{M}) \times \\ &\quad \left(\left[\sum_{m=1}^{\infty} J_m(e_t m) J'_{j+m}(e_t(j+m)) \right] - \left[\sum_{m=j+1}^{\infty} J_m(e_t m) J'_{m-j}(e_t(m-j)) \right] \right) \end{aligned}$$

$$\begin{aligned}
& + \left[\sum_{m=1}^{j-1} J_m(e_t m) J'_{j-m}(e_t(j-m)) \right] \Theta(j, 1) + J'_j(e_t j) \Big) \\
& + \epsilon^4 \mathcal{F}_{v-u} \sum_{j=1}^{\infty} \sin j \mathcal{M} \times \\
& \left(\left[\sum_{m=1}^{j-1} -J_m(m e_t) \left(G_{j-m}(e_t(j-m)) + \frac{2J_{j-m}(e_t(j-m))}{j-m} \right) \right] \Theta(j, 1) \right. \\
& + \left[\sum_{m=j+1}^{\infty} J_m(m e_t) \left(G_{m-j}(e_t) + \frac{2J_{m-j}(e_t(m-j))}{j-m} \right) \right] \\
& - \left[\sum_{m=1}^{\infty} J_m(m e_t) \left(G_{j+m}(e_t) - \frac{2J_{j+m}(e_t(j+m))}{j+m} \right) \right] \\
& \left. - G_j(e_t) + \frac{2J_j(j e_t)}{j} \right). \tag{96}
\end{aligned}$$

The reader should keep in mind Appendix C and D of paper 1 where care is taken of products of infinite series of $\sin j \mathcal{M}$ and $\cos j \mathcal{M}$ with arbitrary coefficients α_j and β_j . This consideration is necessary to collect for terms with the same positive frequencies in the above expressions and many more.

Now we proceed with the trigonometrics of mu , $m \in \mathbb{N}$. We know from the symmetry of $\sin mu$ and $\cos mu$, that only $\sin j \mathcal{M}$ or $\cos j \mathcal{M}$ can contribute. Thus, we decompose

$$\sin mu = \sum_{j=1}^{\infty} \bar{\sigma}_j^m \sin j \mathcal{M}, \tag{97}$$

$$\cos mu = \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j \mathcal{M}. \tag{98}$$

The coefficients $\bar{\sigma}_j^m$ and $\bar{\gamma}_j^m$ can be computed using

$$\bar{\gamma}_j^m = \frac{2}{\pi} \int_0^{\pi} \sin mu \sin j \mathcal{M} d\mathcal{M}, \tag{99}$$

$$\bar{\sigma}_j^m = \frac{2}{\pi} \int_0^{\pi} \cos mu \cos j \mathcal{M} d\mathcal{M}. \tag{100}$$

Switching from the integration over $d\mathcal{M}$ to du in the above equations using the 2PN accurate KE and Taylor expanding everything to ϵ^4 , we can perform the integration. One technical – but easy to manage – issue is to re-convert the arguments of the integrals for a simple application of the Bessel integral formula,

$$J_y(x) = \frac{1}{\pi} \int_0^{\pi} du \cos(yu - x \sin u). \tag{101}$$

Appendix B provides the calculation. The results read

$$\sin(mu) = \sum_{n=1}^{\infty} \bar{\sigma}_j^m \sin j \mathcal{M} + \epsilon^4 \sum_{j=1}^{\infty} \bar{\sigma}_j^m \sin j \mathcal{M}, \tag{102}$$

$$\bar{\sigma}_j^m := \frac{m}{j} \{J_{j-m}(j e_t) + J_{j+m}(j e_t)\}, \tag{103}$$

$$\begin{aligned} \frac{4}{\bar{\sigma}_j^m} := & \frac{1}{2} m \sum_{j=1}^{\infty} \alpha_n \left\{ -J_{j+m-n}(e_t(j-n)) + J_{j-m+n}(e_t(j+n)) \right. \\ & \left. - J_{-j+m+n}(e_t(n-j)) + J_{j+m+n}(e_t(j+n)) \right\}, \end{aligned} \quad (104)$$

$$\cos(mu) = \sum_{j=0}^{\infty} \frac{0}{\bar{\gamma}_j^m} \cos j \mathcal{M} + \epsilon^4 \sum_{j=0}^{\infty} \frac{4}{\bar{\gamma}_j^m} \cos j \mathcal{M}, \quad (105)$$

$$\frac{0}{\bar{\gamma}_j^m} := \frac{m}{j} \{ J_{j-m}(j e_t) - J_{j+m}(j e_t) \} \times \Theta(j, 0) + \delta_{m1} \delta_{0j} \left(-\frac{e_t}{2} \right), \quad (106)$$

$$\begin{aligned} \frac{4}{\bar{\gamma}_j^m} := & \frac{1}{2} m \sum_{n=1}^{\infty} \alpha_n \left\{ J_{-j+m-n}(e_t(-(j+n))) + J_{j+m-n}(e_t(j-n)) \right. \\ & \left. - J_{-j+m+n}(e_t(n-j)) - J_{j+m+n}(e_t(j+n)) \right\} \times \Theta(j, 0) \\ & + \delta_{m1} \delta_{0j} \left(-\frac{\epsilon^2 \left(\sqrt{1 - e_t^2} - 1 \right) (F_{v-u} + F_v)}{e_t} \right). \end{aligned} \quad (107)$$

This prescription is valid in both ADM and harmonic coordinates, where, of course, the appropriate values always have to be included.

5.2 $A(u)^{-n}$ and $A(u)^{-n} \sin u$ as Fourier-Bessel series

We keep expression (A10) from [1],

$$A(u)^{-n} = 1 + b_0^{(n)} + \sum_{j=1}^{\infty} b_j^{(n)} \cos ju, \quad (108)$$

$$\frac{\sin u}{A(u)^n} = \sum_{j=1}^{\infty} S_j^{(n)} \sin ju. \quad (109)$$

In the above two Equations, $b_j^{(n)}$ and $S_j^{(n)}$ are expansion coefficients for harmonics of u . From Equations (102) and (105) we learn that the Fourier-Bessel representation of the above two series get 2PN corrections,¹

$$\begin{aligned} A(u)^{-n} = & 1 + b_0^{(n)} + \sum_{m=1}^{\infty} \bar{\gamma}_0^m b_m^{(n)} + \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \bar{\gamma}_j^k b_k^{(n)} \right) \cos j \mathcal{M} \\ = & \sum_{j=0}^{\infty} \mathcal{A}_j^{(n)} \cos j \mathcal{M}, \end{aligned} \quad (110)$$

$$\begin{aligned} \sin u A(u)^{-n} = & \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \bar{\sigma}_j^k b_k^{(n)} \right) \sin j \mathcal{M} \\ = & \sum_{j=1}^{\infty} \mathcal{S}_j^{(n)} \sin j \mathcal{M}, \end{aligned} \quad (111)$$

$$\mathcal{A}_j^{(n)} = \delta_{j0} \left(1 + b_0^{(n)} + \sum_{m=1}^{\infty} \left[\frac{0}{\bar{\gamma}_0^m} + (\epsilon^4) \frac{4}{\bar{\gamma}_0^m} \right] b_m^{(n)} \right) + \sum_{k=1}^{\infty} \left[\frac{0}{\bar{\gamma}_j^k} + (\epsilon^4) \frac{4}{\bar{\gamma}_j^k} \right] b_k^{(n)}, \quad (112)$$

¹Please recognise the misprint in Equation (39) in paper 1: between $b_0^{(n)}$ and the round bracket, there should be a “+” instead of a “.”.

$$\mathcal{S}_j^{(n)} = \sum_{k=1}^{\infty} \left[\frac{0}{\bar{\sigma}_j^k} + (\epsilon^4) \frac{4}{\bar{\sigma}_j^k} \right] b_k^{(n)}. \quad (113)$$

5.3 Computing one more relevant double series

We have seen in the 1PN case that we have to deal with products of sine-sine and sine-cosine series. At 2PN accuracy, we additionally observe products of cosine-cosine series. The old double series formulas are valid irrespective of the PN order, but we supply the computation of cosine-cosine series products below. Suppose a term of the form

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} \cos k \mathcal{M} \right) \left(\sum_{m=1}^{\infty} B_m \cos m \mathcal{M} \right) \\ = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mathcal{A}_k^{(n)} B_m (\cos [(k-m)\mathcal{M}] + \cos [(k+m)\mathcal{M}]) . \end{aligned} \quad (114)$$

Collecting for contributions with the same the frequency (for $k-m = \pm j$ we obtain $m = k \mp j$ and for part 2, $k+m = j$ we obtain $m = j-k$), the result reads

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (\cos [(k-m)\mathcal{M}] + \cos [(k+m)\mathcal{M}]) \\ = \frac{1}{2} \left(\sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} B_k + \sum_{j=1}^{\infty} \cos j \mathcal{M} \times \left[\sum_{k=j+1}^{\infty} \mathcal{A}_k^{(n)} B_{k-j} + \sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} B_{k+j} + \sum_{k=1}^{j-1} \mathcal{A}_k^{(n)} B_{j-k} \Theta(j, 1) \right] \right) \\ =: \sum_{j=0}^{\infty} P_j^{CC,[n]} \cos j \mathcal{M} . \end{aligned} \quad (115)$$

5.4 Decomposition of $\exp\{-im\phi\}$

What we have done at 1PN accuracy has to be extended to 2PN, especially at the orbital dynamics. It is helpful to find a special decomposition of $e^{-im\phi}$ in such a way that the mode decomposition of any 1PN function of u (to be performed exactly) is not required at this point of calculation. We will combine the terms in such a way that we can use results known from the previous sections,

$$(\phi - \phi_0) = (1 + \mathcal{K}) v + \epsilon^4 (f_{4\phi} \sin 2v + g_{4\phi} \sin 3v) , \quad (116)$$

$$\begin{aligned} v = v(u) &\stackrel{1}{=} 2 \arctan \left\{ \sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan \left[\frac{u}{2} \right] \right\} \\ &\stackrel{2}{=} v_N + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}} \\ &\stackrel{3}{=} (\mathcal{M} + \tilde{v}_N) + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}} , \\ e^{-im(\phi-\phi_0)} &\stackrel{4}{=} e^{-im} [(1+\mathcal{K})v + (\epsilon^4 f_{4\phi}) \sin 2v + (\epsilon^4 g_{4\phi}) \sin 3v] \\ &\stackrel{5}{=} e^{-im} v e^{-im(\epsilon^2 \mathcal{K}_{1\text{PN}} + \epsilon^4 \mathcal{K}_{2\text{PN}})} [v_N + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}}] e^{-im\epsilon^4 [f_{4\phi} \sin 2v + g_{4\phi} \sin 3v]} \\ &\stackrel{6}{=} e^{-im} v e^{-imv_N \epsilon^2 (\mathcal{K}_{1\text{PN}} + \mathcal{K}_{2\text{PN}} \epsilon^2)} e^{-im\epsilon^4 [\mathcal{K}_{1\text{PN}} v_{1\text{PN}} + f_{4\phi} \sin 2v + g_{4\phi} \sin 3v]} \\ &\stackrel{7}{=} e^{-im(\epsilon^2 \mathcal{K}_{1\text{PN}} + \epsilon^4 \mathcal{K}_{2\text{PN}})} \mathcal{M} \times \\ &\quad \underbrace{\underbrace{e^{-imv}}_{\text{Part}_A(m)} \left[1 - im\mathcal{K}_{1\text{PN}} \tilde{v}_N \epsilon^2 - \frac{1}{2} m \tilde{v}_N \epsilon^4 (m\mathcal{K}_{1\text{PN}}^2 \tilde{v}_N + 2i\mathcal{K}_{2\text{PN}}) \right]}_{\text{Part}_B(m)}} \times \end{aligned} \quad (117)$$

$$\underbrace{[1 - im\epsilon^4(\mathcal{K}_{1\text{PN}}v_{1\text{PN}} + f_{4\phi}\sin(2v_{\text{N}}) + g_{4\phi}\sin(3v_{\text{N}}))]}_{\text{Part}_{\text{C}}(m)}] . \quad (118)$$

In step 3, v_{N} decomposes into a purely secular part, namely \mathcal{M} , and a purely oscillatory one with zero average over the interval $\mathcal{M} = [0, 2\pi]$, viz. \tilde{v}_{N} [see Equation (86)]. Therefore, in step 7, the middle term in edgy brackets (“Part_B”) can be written in terms of single and double summations of terms in the right hand side of Equation (86), almost without computational cost. Part_A and Part_C will contribute $A(u)^{-n}$ and also $A(u)^{-n} \sin u$ terms that will multiply with the series of Part_B. These contributions are lengthy and we again skip explicit expressions. In principle, other decompositions are valid as well, but we find it convenient to choose the above one because of its structural clearness. Part_B is essential and we give it explicitly,²

$$\begin{aligned} \text{Part}_{\text{B}}(m) &:= \left[1 - im\mathcal{K}_{1\text{PN}}\tilde{v}_{\text{N}}\epsilon^2 - \frac{1}{2}m\tilde{v}_{\text{N}}\epsilon^4 (m\mathcal{K}_{1\text{PN}}^2\tilde{v}_{\text{N}} + 2i\mathcal{K}_{2\text{PN}}) \right] \\ &= 1 - im\epsilon^2\mathcal{K}_{1\text{PN}} \left[\sum_{j=1}^{\infty} \sin(j\mathcal{M}) G_j(e_t) \right] \\ &\quad - \epsilon^4 \frac{1}{4}m \left\{ \left[\sum_{j=1}^{\infty} m\mathcal{K}_{1\text{PN}}^2 G_j(e_t)^2 \right] \right. \\ &\quad + \sum_{k=1}^{\infty} \left[m\mathcal{K}_{1\text{PN}}^2 \cos(k\mathcal{M}) \times \left(\left[\sum_{j=k+1}^{\infty} G_j(e_t) G_{j-k}(e_t) \right] \right. \right. \\ &\quad \left. \left. - \left[\sum_{j=1}^{k-1} G_j(e_t) G_{k-j}(e_t) \right] \times \Theta(k, 1) \right. \right. \\ &\quad \left. \left. + \left[\sum_{j=1}^{\infty} G_j(e_t) G_{j+k}(e_t) \right] \right) \right] \\ &\quad \left. + 4i\mathcal{K}_{2\text{PN}} \sin(k\mathcal{M}) G_k(e_t) \right] \Bigg\}_2^1 . \quad (119) \end{aligned}$$

For a clear understanding, we sometimes have added auxiliary indices to the brackets. This simply helps to see how deep the bracket in the current expression is. To 1PN order we recognize what we computed in paper 1, Section IV. We will face products of Part_B with powers of $A(u)$. They can be put into the form

$$\begin{aligned} \frac{\text{Part}_{\text{B}}(q)}{A(u)^n} &= \sum_{j=1}^{\infty} \frac{1}{2} iq\epsilon^2 (\mathcal{K}_{1\text{PN}} + \mathcal{K}_{2\text{PN}}\epsilon^2) \sin(j\mathcal{M}) \times \\ &\quad \left\{ \sum_{m=1}^{j-1} \left[-\mathcal{A}_m^{(n)} \Theta(j, 1) G_{j-m}(e_t) \right] + \sum_{m=j+1}^{\infty} \left[\mathcal{A}_m^{(n)} G_{m-j}(e_t) \right] \right. \\ &\quad \left. + \left[\sum_{m=1}^{\infty} -\mathcal{A}_m^{(n)} G_{j+m}(e_t) \right] - 2\mathcal{A}_0^{(n)} G_j(e_t) \right\} + \\ &\quad \hline -\frac{1}{8} \sum_{j=1}^{\infty} \cos(j\mathcal{M}) \left\{ \mathcal{K}_{1\text{PN}}^2 q^2 \epsilon^4 \right\}_1^1 \end{aligned}$$

²In Equation (C5) of paper 1, the very last term should get a factor $\Theta(j, 1)$ to make it consistent with our notation and for convenience of the reader.

$$\begin{aligned}
& \sum_{k=j+1}^{\infty} \left[\mathcal{A}_{k-j}^{(n)} \left(- \sum_{m=1}^{k-1} [G_m(e_t) G_{k-m}(e_t)] + \sum_{n=k+1}^{\infty} [G_n(e_t) G_{n-k}(e_t)] \right. \right. \\
& \left. \left. + \sum_{n=1}^k [G_n(e_t) G_{k+n}(e_t)] \right) \right] \\
& + \sum_{k=1}^{\infty} \left[\mathcal{A}_{j+k}^{(n)} \left(- \sum_{m=1}^{k-1} [G_m(e_t) G_{k-m}(e_t)] + \sum_{n=k+1}^{\infty} [G_n(e_t) G_{n-k}(e_t)] \right. \right. \\
& \left. \left. + \sum_{n=1}^k [G_n(e_t) G_{k+n}(e_t)] \right) \right] \\
& + \Theta(j, 1) \sum_{s=1}^{j-1} \left[\mathcal{A}_{j-s}^{(n)} \left(- \sum_{m=1}^{s-1} [G_m(e_t) G_{s-m}(e_t)] + \sum_{n=s+1}^{\infty} [G_n(e_t) G_{n-s}(e_t)] \right. \right. \\
& \left. \left. + \sum_{n=1}^s [G_n(e_t) G_{n+s}(e_t)] \right) \right] \\
& + 2\mathcal{A}_0^{(n)} \left(- \sum_{m=1}^{j-1} [G_m(e_t) G_{j-m}(e_t)] + \sum_{n=j+1}^{\infty} [G_n(e_t) G_{n-j}(e_t)] \right. \\
& \left. + \left[\sum_{n=1}^j G_n(e_t) G_{j+n}(e_t) \right] \right) \Bigg]_1 + 2\mathcal{A}_j^{(n)} \left(\mathcal{K}_{\text{IPN}}^2 q^2 \epsilon^4 \left[\sum_{k=1}^{\infty} G_k(e_t)^2 \right] - 4 \right) \Bigg]_1 \\
& + \left\{ \sum_{j=1}^{\infty} \left[\frac{1}{8} \mathcal{K}_{\text{IPN}}^2 q^2 \epsilon^4 \left(\mathcal{A}_j^{(n)} \left(- \sum_{k=j+1}^{\infty} [G_k(e_t) G_{k-j}(e_t)] \right. \right. \right. \right. \\
& \left. \left. + \sum_{k=1}^{j-1} [G_k(e_t) G_{j-k}(e_t)] - \sum_{k=1}^j [G_k(e_t) G_{j+k}(e_t)] \right) \right. \right. \\
& \left. \left. - 2\mathcal{A}_0^{(n)} G_j(e_t)^2 \right) \right] + \mathcal{A}_0^{(n)} \right\}. \tag{120}
\end{aligned}$$

The part including $\sin u$ reads

$$\begin{aligned}
\frac{\text{Part}_B(q)}{A(u)^n} \sin u &= \sum_{j=1}^{\infty} \frac{1}{2} \sin(j\mathcal{M}) \times \\
& \left\{ 2\mathcal{S}_j^{(n)} + \sum_{m=1}^{j-1} \left[-\frac{1}{4} \mathcal{K}_{\text{IPN}}^2 q^2 \epsilon^4 \Theta(j, 1) \mathcal{S}_{j-m}^{(n)} \left(\sum_{k=m+1}^{\infty} [G_k(e_t) G_{k-m}(e_t)] \right. \right. \right. \\
& \left. \left. + \sum_{k=1}^{m-1} [-G_k(e_t) G_{m-k}(e_t)] + \sum_{k=1}^m [G_k(e_t) G_{k+m}(e_t)] \right) \right] \\
& + \sum_{m=j+1}^{\infty} \left[\frac{1}{4} \mathcal{K}_{\text{IPN}}^2 q^2 \epsilon^4 \mathcal{S}_{m-j}^{(n)} \left(\sum_{k=m+1}^{\infty} [G_k(e_t) G_{k-m}(e_t)] \right. \right. \\
& \left. \left. + \sum_{k=1}^{m-1} [-G_k(e_t) G_{m-k}(e_t)] + \sum_{k=1}^m [G_k(e_t) G_{k+m}(e_t)] \right) \right] \\
& \left. + \sum_{m=1}^{\infty} \left[-\frac{1}{4} \mathcal{K}_{\text{IPN}}^2 q^2 \epsilon^4 \mathcal{S}_{j+m}^{(n)} \left(\left[\sum_{k=m+1}^{\infty} G_k(e_t) G_{k-m}(e_t) \right] \right. \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\sum_{k=1}^{m-1} -G_k(e_t) G_{m-k}(e_t) \right] + \left[\sum_{k=1}^m G_k(e_t) G_{k+m}(e_t) \right] \Big) \Big) \Big\} \\
& + \left[\sum_{j=1}^{\infty} G_j(e_t)^2 \right] \left[\sum_{j=1}^{\infty} -\frac{1}{4} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \mathcal{S}_j^{(n)} \sin(jM) \right] \\
& - \frac{1}{2} i q \epsilon^2 (\mathcal{K}_{\text{1PN}} + \mathcal{K}_{\text{2PN}} \epsilon^2) \sum_{j=1}^{\infty} \cos(jM) \times \\
& \left\{ \left(\sum_{m=j+1}^{\infty} [\mathcal{S}_{m-j}^{(n)} G_m(e_t)] + \sum_{m=1}^{\infty} [\mathcal{S}_{j+m}^{(n)} G_m(e_t)] \right) \right. \\
& \left. - \Theta(j, 1) \sum_{m=1}^{j-1} [\mathcal{S}_{j-m}^{(n)} G_m(e_t)] \right\} \\
& - \frac{1}{2} i q \epsilon^2 (\mathcal{K}_{\text{1PN}} + \mathcal{K}_{\text{2PN}} \epsilon^2) \left[\sum_{j=1}^{\infty} \mathcal{S}_j^{(n)} G_j(e_t) \right]. \tag{121}
\end{aligned}$$

This will be shortened by writing

$$\frac{\text{Part}_B(q)}{A(u)^n} = \left\{ \sum_{j=0}^{\infty} (\bar{\mathcal{A}}_c)_j^{[n,q]} \cos jM + \sum_{j=1}^{\infty} (\bar{\mathcal{A}}_s)_j^{[n,q]} \sin jM \right\}, \tag{122}$$

$$\sin u \frac{\text{Part}_B(q)}{A(u)^n} = \left\{ \sum_{j=0}^{\infty} (\bar{\mathcal{S}}_c)_j^{[n,q]} \cos jM + \sum_{j=1}^{\infty} (\bar{\mathcal{S}}_s)_j^{[n,q]} \sin jM \right\}, \tag{123}$$

with $(\bar{\mathcal{A}}_c)^{[n,q]}$, $(\bar{\mathcal{A}}_s)^{[n,q]}$, $(\bar{\mathcal{S}}_c)^{[n,q]}$ and $(\bar{\mathcal{S}}_s)^{[n,q]}$ to be easily extracted from Equations (120) and (121) and as well remembering that $\mathcal{A}_j^{(n)}$ and $\mathcal{S}_j^{(n)}$ have 2PN terms. Equations (120) and (121) appear to be relatively complicated, but they are simply expanded applications of the product formulas for sin and cos series, Equations (C1) and (D1) of paper 1, and our current Equation (114). As in paper 1, we can now decompose the wave form into the above irreducible components, from whose we have extracted the time Fourier series representation. To simplify matters, we introduce

$$\frac{\text{Part}_B(q)}{A(u)^n} =: \mathcal{F}_{[nq]}(u), \tag{124}$$

$$\sin u \frac{\text{Part}_B(q)}{A(u)^n} =: \mathcal{F}_{S[nq]}(u). \tag{125}$$

5.5 Multipole moment decomposition: a brief posting of the results

Using the results of the previous section, we decompose the multipole coefficients I and S as

$$I_{am}^{(a)} = \mathcal{C}_{I;am} e^{-miK\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_k^{a-2} \alpha_{[km]} \mathcal{F}_{[km]}(u) + \sum_{k'}^{a-2} \tilde{\alpha}_{[km]} \mathcal{F}_{S[k'm]}(u) \right\}, \tag{126}$$

$$S_{bm}^{(b)} = \mathcal{C}_{S;bm} e^{-miK\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_k^{b-1} \beta_{[km]} \mathcal{F}_{[km]}(u) + \sum_{k'}^{b-1} \tilde{\beta}_{[km]} \mathcal{F}_{S[k'm]}(u) \right\}, \tag{127}$$

symbolically, where $\mathcal{C}_{I;am}$ and $\mathcal{C}_{S;am}$ are some pre-factors and $\alpha, \tilde{\alpha}, \beta$ and $\tilde{\beta}$ are coefficients to be determined, k and k' are some summation dummy indices with boundaries depending on a and m

and depending on the type (I or S), and $a - 2$ and $b - 1$ are labels for $\alpha, \tilde{\alpha}$ and for $\beta, \tilde{\beta}$, counting the order of c^{-1} . The associated components of I and S , including Part_A and Part_C , are given in Appendix D. Some “pre – Fourier” domain reads

$$I_{am}^{(a)} = \mathcal{C}_{I;am} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_j \sin j\mathcal{M} I_{Sj}^{(a)} + \sum_j \cos j\mathcal{M} I_{Cj}^{(a)} \right\}, \quad (128)$$

$$S_{bm}^{(b)} = \mathcal{C}_{S;bm} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_j \sin j\mathcal{M} S_{Sj}^{(b)} + \sum_j \cos j\mathcal{M} S_{Cj}^{(b)} \right\}, \quad (129)$$

with

$$I_{Sj}^{(a)} := \left(\sum_k^{a-2} \alpha_{[km]} (\bar{\mathcal{A}}_s)_j^{[k,m]} + \sum_{k'}^{a-2} \tilde{\alpha}_{[k'm]} (\bar{\mathcal{S}}_s)_j^{[k',m]} \right), \quad (130)$$

$$I_{Cj}^{(a)} := \left(\sum_k^{a-2} \alpha_{[km]} (\bar{\mathcal{A}}_c)_j^{[k,m]} + \sum_{k'}^{a-2} \tilde{\alpha}_{[k'm]} (\bar{\mathcal{S}}_c)_j^{[k',m]} \right), \quad (131)$$

$$S_{Sj}^{(b)} := \left(\sum_k^{b-1} \beta_{[km]} (\bar{\mathcal{A}}_s)_j^{[k,m]} + \sum_{k'}^{b-1} \tilde{\beta}_{[k'm]} (\bar{\mathcal{S}}_s)_j^{[k',m]} \right), \quad (132)$$

$$S_{Cj}^{(b)} := \left(\sum_k^{b-1} \beta_{[km]} (\bar{\mathcal{A}}_c)_j^{[k,m]} + \sum_{k'}^{b-1} \tilde{\beta}_{[k'm]} (\bar{\mathcal{S}}_c)_j^{[k',m]} \right), \quad (133)$$

for $a \in [2, 6], b \in [2, 5]$,

and for extracting the pure Fourier domain representation with delta distributions – (and not the one mixed in exponential and trigonometric representation as in Equations (128) and (129)) – we take the Fourier transformation of the $\sin j\mathcal{M}$ and $\cos j\mathcal{M}$ terms,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-im\mathcal{K}\mathcal{M}} \sin j\mathcal{M} e^{i\omega t} dt = i\sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} + \mathcal{K}m\mathcal{N} - \omega) - i\sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} - \mathcal{K}m\mathcal{N} + \omega) \quad (134)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-im\mathcal{K}\mathcal{M}} \cos j\mathcal{M} e^{i\omega t} dt = \sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} + \mathcal{K}m\mathcal{N} - \omega) + \sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} - \mathcal{K}m\mathcal{N} + \omega). \quad (135)$$

This is valid only in the purely conservative orbital dynamics case.

5.6 The effect of radiation reaction

Up to now, we have used only the conservative parameterization of Section 2 hitherto. When we include the slow-in-time variation of the orbital elements due to radiation reaction, the Equations (128) and (129) are no longer valid, because the radial frequency \mathcal{N} is no longer a constant. In paper 1, we gave a detailed overview over what to do when eccentric GW signals with chirp have to be Fourier transformed using the algorithm of the steepest decent or equivalently called “the method of the stationary phase”. The only difference to be recognized is that we may include 2PN accurate formulas for \dot{e}_t and $\dot{\mathcal{N}}$ and the QKP, but the analytic integration of those EOM will be skipped because of the extensive space filling of the results as we have faced already at 1PN. A future publication may extend the calculation of Section VI of paper 1 with the help of Appell’s integral formula (see e.g. [32]), the *orbital averaged* 2PN accurate EOM

$$\left\langle \frac{d\mathcal{N}}{dt} \right\rangle_{\text{orbit}} = \mathfrak{N}_{2\text{PN}}(\mathcal{N}, e_t), \quad (136)$$

$$\left\langle \frac{de_t}{dt} \right\rangle_{\text{orbit}} = \mathfrak{E}_{2\text{PN}}(\mathcal{N}, e_t), \quad (137)$$

to be taken from the expressions (4.32) and (4.34) of reference [22] and converted appropriately to \mathcal{N} and e_t , and the perturbation routines for inverting the formulas for the time of steepest decent we are familiar with.

The interested reader may find the stationary phase approximation (SPA) results for circular orbits in various data analysis papers [33, 34, 35, 36].

5.7 Considerations about limiting of the series

Infinite series have to be restricted to finite ones for practical issues. In paper 1 we gave instructions how to limit the 1PN series as we set up the following properties concerning the orders of involved terms:

1. Bessel functions of order n : $J_n(ne_t) \sim \mathcal{O}(e_t^n)$,
2. The $v(u)$ expansion coefficients: $G_n(ne_t) \sim \mathcal{O}(e_t^n)$,
3. Even-in- u expansion coefficients: $\mathcal{A}_j^{(n)} \sim \mathcal{O}(e_t^j)$,
4. Odd-in- u expansion coefficients: $\mathcal{S}_j^{(n)} \sim \mathcal{O}(e_t^{j-1})$,
5. The $\sin v$ expansion: $J'_n(e_t) = \frac{1}{2}(J_{n-1}(e_t) - J_{n+1}(e_t)) = \mathcal{O}(e_t^{n-1})$,
6. Double series expansions: $P_j^{CS,[n]} \sim \mathcal{O}(e_t^j)$ and $P_j^{SS,[n]} \sim \mathcal{O}(e_t^{j-1})$.

These computations are still valid at 2PN and have to be applied to each series where $e_t^{\text{sum index}}$ plays a role. The result of this is obvious but lengthy, so we skip the provision. The interested reader may take a look at [30] for more information on an estimate of the error when using finite sums. Having double, triple, \dots , maximally n -tuples of summations, each evaluated up to some order $\mathcal{O}(e_t^{\mathfrak{M}})$ and, thus, containing \mathfrak{M} terms (plus or minus some finite number), we have a computational cost of $\sim \mathfrak{M}^n$ terms per each time step \mathcal{M}_k and $\text{NSP} \times \mathfrak{M}^n$ term computations in total, where NSP is the number of sampling points, $k = (0, \dots, \text{NSP} - 1)$. In our case, a typical value could be $n = 5$.

6 Conclusion

In this article we provide 2PN accurate GW forms in terms of tensor spherical harmonics. The analytic Fourier-Bessel series of the inverted KE as well as the sines and cosines of the eccentric anomaly have been evaluated. These series may be interesting for perturbation theory of celestial mechanics. We could verify parts of the results of [20] and reproduce results of the ideas of paper 1.

One slight drawback should be mentioned. Without appropriate optimizing, the presented routine is far less than quickly computable, for example for a data rate of 4096 points per second and several minutes to be observed and a restriction to errors of, say, $< \mathcal{O}(e_t^5)$. A first numerical insight, done in C for the 1PN case, showed a CPU time of \sim one minute for the case of 128 data points at an error of $\mathcal{O}(e_t^6)$. It can, for example, give an impression of the orders of magnitude of how many harmonics may have to be included for a data analysis investigation. Its CPU time consumption should be improved by atomising the series computation to make it attractive for researchers in data analysis.

A future investigation may include the 3PN GW amplitude [29] and the 3PN QKP for point particles without spin, or even more the spin dependent multipole moments of [37, 38, 39, 40] and [41] using the QKP for aligned spinning compact binaries and a well-suited and optimized numerical implementation.

Acknowledgements We thank Jan Sperrhake for useful discussions. MT likes to thank Johannes Hartung for an enlightening introduction on automatise and efficient programming of *Mathematica*, in which we coded most of our work. This work was funded by the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 “Gravitationswellenastronomie” and the DLR through “LISA – Germany”.

A The solution to the 2PN accurate Kepler Equation: an alternative way to derive it and a useful check

The 2PN accurate KE,

$$\begin{aligned}\mathcal{M} &= u - e_t \sin u + \epsilon^4 (\mathcal{F}_{v-u}(v - u) + \mathcal{F}_v \sin v) \\ &=: u - e_t \sin u + \epsilon^4 \mathcal{F}_4(u),\end{aligned}\tag{138}$$

can be inverted by defining

$$g(\mathcal{M} - \epsilon^4 \mathcal{F}_4(u)) = u.\tag{139}$$

as the appropriate solution, however it will look like. We can Taylor expand it around the Newtonian solution,

$$g_N(\mathcal{M}) - g'_N(\mathcal{M}) \epsilon^4 \mathcal{F}_4(u) = u.\tag{140}$$

The Newtonian solution g is known,

$$g_N(\mathcal{M}) = \mathcal{M} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n e_t) \sin n\mathcal{M},\tag{141}$$

and (140) reads

$$u(\mathcal{M}) = g_N(\mathcal{M}) - \left(1 + \sum_{n=1}^{\infty} 2J_n(n e_t) \cos n\mathcal{M}\right) \epsilon^4 \mathcal{F}_4(g_N(\mathcal{M})).\tag{142}$$

We know

$$v - u = \sum_{n=1}^{\infty} \left[G_n(n e_t) - \frac{2}{n} J_n(n e_t) \right] \sin n\mathcal{M},\tag{143}$$

$$\sin v = \frac{\sqrt{1 - e_t^2} \sin u}{1 - e_t \cos u} = \sqrt{1 - e_t^2} 2 \sum_{n=1}^{\infty} J'_n(n e_t) \sin n\mathcal{M}.\tag{144}$$

Inserting (143) and (144) into (142) and applying the product rule for a sin and a cos series, we obtain the result of (96). This is a nice calculation, so let us show it in detail.

We abbreviate

$$\mathcal{F}_4(g_N(\mathcal{M})) = \sum_{n=1}^{\infty} \alpha_n \sin n\mathcal{M},\tag{145}$$

and read the coefficients α from Equations (143) and (144). In fact, for the comparison, their form does not matter. Equation (142) together with the rule for products of a sin and a cos series reads

$$u = \sum_{n=1}^{\infty} \frac{J_n(n e_t)}{n} \sin n\mathcal{M}$$

$$-\epsilon^4 \sum_{n=1}^{\infty} \left(\alpha_n + 2P_n^{CS,[n]}[J_k(ke_t); \alpha_k] \right) \sin n\mathcal{M}, \quad (146)$$

$$\begin{aligned} 2P_n^{CS,[n]}[J_k(ke_t); \alpha_k] &\stackrel{1}{=} \left\{ \Theta(n, 1) \sum_{k=1}^{n-1} \alpha_{n-k} J_k(ke_t) + \sum_{k=1}^{\infty} \alpha_{n+k} J_k(ke_t) - \sum_{k=n+1}^{\infty} \alpha_{k-n} J_k(ke_t) \right\} \\ &\stackrel{2}{=} \sum_{k=1}^{n-1} \alpha_k J_{n-k}((n-k)e_t) + \sum_{k=n+1}^{\infty} \alpha_k J_{n-k}((n-k)e_t) \\ &\quad - \sum_{k=1}^{\infty} \alpha_k J_{n+k}((n+k)e_t). \end{aligned} \quad (147)$$

Θ just dropped out as for $n = 1$, α_k for $k \leq 0$ vanish anyway. In summation with α_k the last line gives $\sum_{k=1}^{\infty} \alpha_k (J_{n-k}((n-k)e_t) - J_{n+k}((n+k)e_t))$, remembering that for the $k = n$ term, $J_0(0) = 1$.

Let us, in contrast, directly derive the expansion coefficients via integration and assume that g is the solution to the KE, $u = g(\mathcal{M})$. Then, at $n\pi$ ($n \in \mathbb{Z}$), there are fixed points of the KE: $n\pi = u = \mathcal{M}$ and $g(\mathcal{M}) - \mathcal{M}$ can be expressed in sin series,

$$g(\mathcal{M}) - \mathcal{M} = \sum_{n=1}^{\infty} A_n \sin(n\mathcal{M}). \quad (148)$$

The expansion coefficients, directly computed via integration read

$$\begin{aligned} A_n &\stackrel{1}{=} \frac{2}{\pi} \int_0^{\pi} [g(\mathcal{M}) - \mathcal{M}] \sin(n\mathcal{M}) d\mathcal{M} \\ &\stackrel{2}{=} -\frac{2}{n\pi} \int_0^{\pi} [g(\mathcal{M}) - \mathcal{M}] d(\cos n\mathcal{M}) \\ &\stackrel{3}{=} [\text{boundary} = 0] + \frac{2}{n\pi} \int_0^{\pi} \cos(n\mathcal{M}) [g'(\mathcal{M}) - 1] d\mathcal{M} \\ &\stackrel{4}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n\mathcal{M}\} d(g(\mathcal{M})) \\ &\stackrel{5}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n(u - e_t \sin u + \epsilon^4 \mathcal{F}_4(u))\} du \\ &\stackrel{6}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n(u - e_t \sin u)\} du - \frac{2}{\pi} \int_0^{\pi} \sin \{n(u - e_t \sin u)\} \epsilon^4 \sum_{k=1}^{\infty} \alpha_k \sin k\mathcal{M} \\ &\stackrel{7}{=} \frac{2J_n(ne_t)}{n} - \epsilon^4 \frac{1}{\pi} \int_0^{\pi} \sum_{k=1}^{\infty} \alpha_k (\cos [(k-n)(u - e_t \sin u)] - \cos [(k+n)(u - e_t \sin u)]) du \\ &\stackrel{8}{=} \frac{2J_n(ne_t)}{n} - \epsilon^4 \sum_{k=1}^{\infty} \alpha_k (J_{k-n}(ne_t) - J_{k+n}(ne_t)), \end{aligned} \quad (149)$$

and we see that the 2PN coefficient shows agreement in both calculations. In step 5 we used Equation (138), in step 6 we Taylor expanded the argument of \cos around the Newtonian \mathcal{M} and in step 7 we used that only Newtonian \mathcal{M} is required in the sum. The trigonometrics of u are dealt with equivalently.

B Fourier representation: $\sin mu$ and $\cos mu$ at 2PN

As an exemplary calculation, we determine the expansion coefficients of $\cos mu$,

$$\cos mu = \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j\mathcal{M}. \quad (150)$$

Using integration by parts, the computation turns out to be

$$\begin{aligned} \bar{\gamma}_j^m &= \frac{2}{\pi} \int_0^\pi \cos mu \cos j\mathcal{M} d\mathcal{M} \\ &= [\cos mu \sin j\mathcal{M}]_0^\pi - \frac{2}{j\pi} \int_0^\pi (d \cos mu) \sin j\mathcal{M} d\mathcal{M} \\ &= [\dots]_0^\pi - \frac{2m}{j\pi} \int_0^\pi (\sin mu) \sin j\mathcal{M} \frac{du}{d\mathcal{M}} d\mathcal{M} \\ &= \frac{m}{j\pi} \int_0^\pi (\cos[mu + j\mathcal{M}_{2\text{PN}}] - \cos[mu - j\mathcal{M}_{2\text{PN}}]) du \\ &= \frac{m}{j\pi} \int_0^\pi (\cos[mu + j\mathcal{M}_\text{N}] - \cos[mu - j\mathcal{M}_\text{N}]) du \\ &\quad - \frac{m}{\pi} \int_0^\pi (\sin[mu + j\mathcal{M}_\text{N}] - \sin[mu - j\mathcal{M}_\text{N}]) \epsilon^4 \sum_{f=1}^{\infty} \alpha_f \sin f\mathcal{M}_\text{N} du \\ &= \frac{m}{j} (J_{j+m}(je_t) - J_{j-m}(je_t)) \\ &\quad - \frac{m}{2\pi} \sum_{n=1}^{\infty} \int_0^\pi \alpha_n [\cos(j\mathcal{M} - mu - \mathcal{M}n) - \cos(j\mathcal{M} - mu + \mathcal{M}n) \\ &\quad - \cos(j\mathcal{M} + mu - \mathcal{M}n) + \cos(j\mathcal{M} + mu + \mathcal{M}n)] du \\ &= \frac{m}{j} (J_{j+m}(je_t) - J_{j-m}(je_t)) \\ &\quad - \sum_{n=1}^{\infty} \frac{m\alpha_n}{2\pi} \int_0^{2\pi} \left\{ \begin{aligned} &\cos[(j-n)(u - e_t \sin(u)) - mu] \\ &- \cos[(j+n)(u - e_t \sin(u)) - mu] \\ &- \cos[(j-n)(u - e_t \sin(u)) + mu] \\ &+ \cos[(j+n)(u - e_t \sin(u)) + mu] \end{aligned} \right\} du. \end{aligned} \quad (151)$$

The task is now to bring these integrals to the form

$$\frac{1}{\pi} \int_0^\pi \cos[\mathbf{x}_i(u - \mathbf{y}_i \sin u)] = J_{\mathbf{x}_i}(\mathbf{x}_i \mathbf{y}_i), \quad (152)$$

with some prefactor \mathbf{x} and “eccentricity” \mathbf{y} to be determined for each special case. In order of appearance in the last equation above, these eccentricities read

$$\mathbf{y}_1 = \frac{e_t(j-n)}{j-m-n}, \quad (153)$$

$$\mathbf{y}_2 = \frac{e_t(j+n)}{j-m+n}, \quad (154)$$

$$\mathbf{y}_3 = \frac{e_t(j-n)}{j+m-n}, \quad (155)$$

$$\mathbf{y}_4 = \frac{e_i(j+n)}{j+m+n}, \quad (156)$$

\mathbf{x}_i is simply the denominator. Herewith, the rest is easy to calculate and so are the coefficients of $\sin mu$.

C The relevant tensor spherical harmonics

We take the definitions from [18], Equations (A1) – (A5) therein,

$$T_{LM}^{(m)} = A_{LM} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) - i B_{LM} (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}), \quad (157)$$

$$T_{LM}^{(e)} = B_{LM} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) - i A_{LM} (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}), \quad (158)$$

$$A_{LM} = 2 C_L \left\{ \frac{\partial^2}{\partial \theta^2} + \frac{L(L+1)}{2} \right\} Y_{LM}(\theta, \phi), \quad (159)$$

$$B_{LM} = -2 C_L \left\{ \frac{\partial}{\partial \theta} - \cot \theta \right\} Y_{LM}(\theta, \phi), \quad (160)$$

$$C_L = [2 L (L+1)(L+2)(L-1)]^{-1/2}, \quad (161)$$

$$T_{LM}^{E2} = T_{LM}^{(m)}, \quad (162)$$

$$T_{LM}^{B2} = -i T_{LM}^{(e)}, \quad (163)$$

with $\hat{\theta}$ and $\hat{\phi}$ being basis unit vectors in θ and ϕ direction (θ is the angle between the orbital angular momentum and the line-of-sight vector \mathbf{N} , and ϕ measures the angle from the x axis to \mathbf{N} projected onto the (x, y) plane, see Figure 3 of [18]). The relevant multipoles read

$$T_{22}^{E2} = \frac{1}{8} \sqrt{\frac{5}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \frac{1}{2} (\cos(2\theta) + 3) + 2i \cos(\theta) (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \right], \quad (164)$$

$$T_{21}^{E2} = \frac{1}{4} \sqrt{\frac{5}{2\pi}} e^{2i\phi} \left[\cos(\theta) \sin(\theta) (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) + i \sin(\theta) (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \right], \quad (165)$$

$$T_{20}^{E2} = \frac{1}{8} \sqrt{\frac{15}{\pi}} \sin^2(\theta) (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}), \quad (166)$$

$$T_{22}^{B2} = \frac{-1}{16} \sqrt{\frac{5}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right], \quad (167)$$

$$T_{21}^{B2} = \frac{1}{4} \sqrt{\frac{5}{2\pi}} i e^{i\phi} \left[(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) + i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) \right] \sin(\theta), \quad (168)$$

$$T_{20}^{B2} = \frac{-1}{8} \sqrt{\frac{15}{\pi}} (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \sin^2(\theta), \quad (169)$$

$$T_{33}^{E2} = -\frac{1}{32} \sqrt{\frac{21}{\pi}} e^{3i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(2\theta) + 3) \right] \sin(\theta), \quad (170)$$

$$T_{32}^{E2} = \frac{1}{32} \sqrt{\frac{7}{2\pi}} e^{2i\phi} \left[8i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(2\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (5 \cos(\theta) + 3 \cos(3\theta)) \right], \quad (171)$$

$$T_{31}^{E2} = \frac{1}{32} \sqrt{\frac{35}{\pi}} e^{i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (172)$$

$$T_{30}^{E2} = \frac{1}{8} \sqrt{\frac{105}{\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \sin^2(\theta), \quad (173)$$

$$T_{33}^{B2} = \frac{1}{32} \sqrt{\frac{21}{\pi}} e^{3i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin(\theta), \quad (174)$$

$$T_{32}^{B2} = \frac{-1}{32} \sqrt{\frac{7}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (5 \cos(\theta) + 3 \cos(3\theta)) - 8i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(2\theta) \right], \quad (175)$$

$$T_{31}^{B2} = \frac{-1}{32} \sqrt{\frac{35}{\pi}} e^{i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (3 \cos(2\theta) + 1) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin(\theta), \quad (176)$$

$$T_{30}^{B2} = \frac{-1}{8} \sqrt{\frac{105}{\pi}} (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) \sin^2(\theta), \quad (177)$$

$$T_{44}^{E2} = \frac{3}{32} \sqrt{\frac{7}{2\pi}} e^{4i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(2\theta) + 3) \right] \sin^2(\theta), \quad (178)$$

$$T_{43}^{E2} = \frac{-3}{32} \sqrt{\frac{7}{\pi}} e^{3i\phi} \left[4 (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos^3(\theta) + i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (179)$$

$$T_{42}^{E2} = \frac{3}{64\sqrt{2\pi}} e^{2i\phi} \left[2i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (\cos(\theta) + 7 \cos(3\theta)) \right. \\ \left. + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (4 \cos(2\theta) + 7 \cos(4\theta) + 5) \right], \quad (180)$$

$$T_{41}^{E2} = \frac{3}{32\sqrt{\pi}} e^{i\phi} \left[i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (7 \cos(2\theta) + 5) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (5 \cos(\theta) + 7 \cos(3\theta)) \right] \sin(\theta), \quad (181)$$

$$T_{40}^{E2} = \frac{3}{32} \sqrt{\frac{5}{\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (7 \cos(2\theta) + 5) \sin^2(\theta), \quad (182)$$

$$T_{44}^{B2} = \frac{-3}{32} \sqrt{\frac{7}{2\pi}} e^{4i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin^2(\theta), \quad (183)$$

$$T_{43}^{B2} = \frac{3}{32} \sqrt{\frac{7}{\pi}} e^{3i\phi} \left[4 (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos^3(\theta) - i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (184)$$

$$T_{42}^{B2} = \frac{-3}{64\sqrt{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (4 \cos(2\theta) + 7 \cos(4\theta) + 5) \right. \\ \left. - 2i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(\theta) + 7 \cos(3\theta)) \right], \quad (185)$$

$$T_{41}^{B2} = \frac{-3}{32\sqrt{\pi}} e^{i\phi} \left[(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (5 \cos(\theta) + 7 \cos(3\theta)) - i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (7 \cos(2\theta) + 5) \right] \sin(\theta), \quad (186)$$

$$T_{40}^{B2} = \frac{-3}{32} \sqrt{\frac{5}{\pi}} (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (7 \cos(2\theta) + 5) \sin^2(\theta), \quad (187)$$

$$T_{55}^{E2} = -\frac{1}{64} \sqrt{\frac{165}{\pi}} e^{5i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(2\theta) + 3) \right] \sin^3(\theta), \quad (188)$$

$$T_{54}^{E2} = \frac{1}{64} \sqrt{\frac{33}{2\pi}} e^{4i\phi} \left[8i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (2 \cos(2\theta) + 1) \right. \\ \left. + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (19 \cos(\theta) + 5 \cos(3\theta)) \right] \sin^2(\theta), \quad (189)$$

$$T_{53}^{E2} = -\frac{1}{256} \sqrt{\frac{33}{\pi}} e^{3i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (7 \cos(\theta) + 9 \cos(3\theta)) \right. \\ \left. + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (28 \cos(2\theta) + 15 \cos(4\theta) + 21) \right] \sin(\theta), \quad (190)$$

$$T_{52}^{E2} = \frac{1}{128} \sqrt{\frac{11}{2\pi}} e^{2i\phi} \left[8i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3 \cos(4\theta)) \right. \\ \left. + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (14 \cos(\theta) + 3(\cos(3\theta) + 5 \cos(5\theta))) \right], \quad (191)$$

$$T_{51}^{E2} = \frac{1}{128} \sqrt{\frac{77}{2\pi}} e^{i\phi} \left[4i (\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}) (5 \cos(\theta) + 3 \cos(3\theta)) \right]$$

$$+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(12\cos(2\theta)+15\cos(4\theta)+5)\Big]\sin(\theta), \quad (192)$$

$$T_{50}^{E2} = \frac{1}{64}\sqrt{\frac{1155}{\pi}}\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)[5\cos(\theta)+3\cos(3\theta)]\sin^2(\theta), \quad (193)$$

$$T_{55}^{B2} = \frac{1}{64}\sqrt{\frac{165}{\pi}}e^{5i\phi}\left[\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(\cos(2\theta)+3)-4i\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)\cos(\theta)\right]\sin^3(\theta), \quad (194)$$

$$T_{54}^{B2} = -\frac{1}{64}\sqrt{\frac{33}{2\pi}}e^{4i\phi}\left[\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(19\cos(\theta)+5\cos(3\theta))\right. \\ \left.-8i\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(2\cos(2\theta)+1)\right]\sin^2(\theta), \quad (195)$$

$$T_{53}^{B2} = \frac{1}{256}\sqrt{\frac{33}{\pi}}e^{3i\phi}\left[\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(28\cos(2\theta)+15\cos(4\theta)+21)\right. \\ \left.-4i\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(7\cos(\theta)+9\cos(3\theta))\right]\sin(\theta), \quad (196)$$

$$T_{52}^{B2} = -\frac{1}{128}\sqrt{\frac{11}{2\pi}}e^{2i\phi}\left[\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(14\cos(\theta)+3(\cos(3\theta)+5\cos(5\theta)))\right. \\ \left.-8i\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(\cos(2\theta)+3\cos(4\theta))\right], \quad (197)$$

$$T_{51}^{B2} = -\frac{1}{128}\sqrt{\frac{77}{2\pi}}e^{i\phi}\left[\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(12\cos(2\theta)+15\cos(4\theta)+5)\right. \\ \left.-4i\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(5\cos(\theta)+3\cos(3\theta))\right]\sin(\theta), \quad (198)$$

$$T_{50}^{B2} = -\frac{1}{64}\sqrt{\frac{1155}{\pi}}\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)[5\cos(\theta)+3\cos(3\theta)]\sin^2(\theta), \quad (199)$$

$$T_{66}^{E2} = \frac{3}{256}\sqrt{\frac{715}{2\pi}}e^{6i\phi}\left[4i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)\cos(\theta)+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(\cos(2\theta)+3)\right]\sin^4(\theta), \quad (200)$$

$$T_{65}^{E2} = -\frac{1}{256}\sqrt{\frac{2145}{2\pi}}e^{5i\phi}\left[2i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(5\cos(2\theta)+3)\right. \\ \left.+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(13\cos(\theta)+3\cos(3\theta))\right]\sin^3(\theta), \quad (201)$$

$$T_{64}^{E2} = \frac{1}{1024}\sqrt{\frac{195}{\pi}}e^{4i\phi}\left[8i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(13\cos(\theta)+11\cos(3\theta))\right. \\ \left.+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(92\cos(2\theta)+33\cos(4\theta)+67)\right]\sin^2(\theta), \quad (202)$$

$$T_{63}^{E2} = -\frac{3}{1024}\sqrt{\frac{13}{2\pi}}e^{3i\phi}\left[2i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(52\cos(2\theta)+55\cos(4\theta)+21)\right. \\ \left.+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(126\cos(\theta)+75\cos(3\theta)+55\cos(5\theta))\right]\sin(\theta), \quad (203)$$

$$T_{62}^{E2} = \frac{1}{4096}\sqrt{\frac{13}{2\pi}}e^{2i\phi}\left[4i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(10\cos(\theta)+81\cos(3\theta)+165\cos(5\theta))\right. \\ \left.+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(289\cos(2\theta)+30(\cos(4\theta)+7)+495\cos(6\theta))\right], \quad (204)$$

$$T_{61}^{E2} = \frac{1}{1024}\sqrt{\frac{65}{\pi}}e^{i\phi}\left[2i\left(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}\right)(60\cos(2\theta)+33\cos(4\theta)+35)\right. \\ \left.+\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)(70\cos(\theta)+87\cos(3\theta)+99\cos(5\theta))\right]\sin(\theta), \quad (205)$$

$$T_{60}^{E2} = \frac{1}{512}\sqrt{\frac{1365}{2\pi}}\left(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}\right)[60\cos(2\theta)+33\cos(4\theta)+35]\sin^2(\theta). \quad (206)$$

D The multipole components in terms of Fourier coefficients

$$\begin{aligned} I^{(2)}_{22} &= -8\sqrt{\frac{2\pi}{5}}|E|\mu e_t^{-2}e^{-2i\mathcal{K}\mathcal{M}}e^{-i2\phi_0} \times \\ &\quad \left\{ \sum_{k=0}^7 {}^0\alpha_{[k2]}\mathcal{F}_{[k2]}(u) + \sum_{k=1}^7 {}^0\tilde{\alpha}_{[k2]}\mathcal{F}_{S[k2]}(u) \right\}, \end{aligned} \quad (207)$$

$$\begin{aligned} {}^0\alpha_{[02]} &:= [2 - e_t^2] + \epsilon^2 \frac{1}{14}|E|[(9\eta - 3)e_t^2 + 94\eta - 442] + \epsilon^4 \frac{-|E|^2}{42\mathcal{W}_{e_t}^2}[(\eta(11\eta + 16) - 5)e_t^4 \\ &\quad + 3(\eta(197\eta - 1216) + 3717)e_t^2 + 32(40 - 63\mathcal{W}_{e_t})\eta + 5040\mathcal{W}_{e_t} - 602\eta^2 + 2798], \\ {}^0\alpha_{[12]} &:= [e_t^2 - 2] + \epsilon^2 \frac{1}{14}|E|((3 - 9\eta)e_t^2 - 94\eta + 778) + \epsilon^4 \frac{-|E|^2}{42\mathcal{W}_{e_t}^2}[(5 - \eta(11\eta + 16))e_t^4 \\ &\quad + 2\eta(1512\mathcal{W}_{e_t} + 301\eta - 64) - 3e_t^2(\eta(168\mathcal{W}_{e_t} + 197\eta - 3184) - 420\mathcal{W}_{e_t} + 10029) \\ &\quad - 7560\mathcal{W}_{e_t} + 2530], \end{aligned} \quad (208)$$

$$\begin{aligned} {}^0\alpha_{[22]} &:= -2(\mathcal{W}_{e_t}^2) + \epsilon^2 \frac{1}{21}|E|(-9\eta(e_t^2 + 17) - 151e_t^2 + 163) + \epsilon^4 - \frac{|E|^2}{126\mathcal{W}_{e_t}^2} \left[e_t^2(11340\mathcal{W}_{e_t} \right. \\ &\quad + 45322 - \eta(4536\mathcal{W}_{e_t} + 1123\eta + 15623)) + (3(951 - 119\eta)\eta + 4188)e_t^4 \\ &\quad \left. + 2\eta(4536\mathcal{W}_{e_t} + 740\eta - 2435) - 22680\mathcal{W}_{e_t} - 57574 \right], \end{aligned} \quad (209)$$

$$\begin{aligned} {}^0\alpha_{[32]} &:= 2[e_t^2 - 1]^2 + \epsilon^2|E|\left[\frac{1}{21}\mathcal{W}_{e_t}^2(11(3\eta + 13)e_t^2 + 129\eta - 1163)\right] + \epsilon^4 \frac{|E|^2}{189} \left[\eta(31752\mathcal{W}_{e_t} \right. \\ &\quad + 3433\eta + 13700) - 79380\mathcal{W}_{e_t} - 60259 + e_t^2(\eta(-4536\mathcal{W}_{e_t} - 3481\eta + 10081) \\ &\quad \left. + 11340\mathcal{W}_{e_t} - 93512) + 3((830 - 47\eta)\eta + 2117)e_t^4 \right], \end{aligned} \quad (210)$$

$$\begin{aligned} {}^0\alpha_{[42]} &:= \epsilon^2|E|\left[\frac{4}{21}|E|\mathcal{W}_{e_t}^4(15\eta + 121)\right] + \epsilon^4 \frac{-|E|^2}{378}\mathcal{W}_{e_t}^2 \left[9(\eta(11\eta + 3339) - 9453)e_t^2 \right. \\ &\quad \left. + 4\eta(6804\mathcal{W}_{e_t} + 1211\eta + 7423) - 68040\mathcal{W}_{e_t} + 77866 \right], \end{aligned} \quad (211)$$

$$\begin{aligned} {}^0\alpha_{[52]} &:= \epsilon^2|E|\left[\frac{-4}{7}(3\eta - 1)\mathcal{W}_{e_t}^6\right] + \epsilon^4 \frac{|E|^2}{378}\mathcal{W}_{e_t}^4 \left[3(4\eta(349\eta + 917) - 1117)e_t^2 \right. \\ &\quad \left. - 16\eta(308\eta + 1891) + 95270 \right], \end{aligned}$$

$${}^0\alpha_{[62]} := \epsilon^4 \frac{|E|^2}{189}\mathcal{W}_{e_t}^6(\eta(6853\eta + 20333) - 12793), \quad (212)$$

$${}^0\alpha_{[72]} := -\epsilon^4 \frac{5}{63}|E|^2\mathcal{W}_{e_t}^8(4\eta(53\eta + 103) - 131), \quad (213)$$

$$\begin{aligned} {}^0\tilde{\alpha}_{[12]} &:= i2\mathcal{W}_{e_t}e_t + i\frac{|E|\epsilon^2e_t((109 - 19\eta)e_t^2 + 47\eta - 221)}{7\mathcal{W}_{e_t}} \\ &\quad + i\epsilon^4 \frac{|E|^2}{21\mathcal{W}_{e_t}^6} \left[-252\mathcal{W}_{e_t}^4(2\eta - 5)e_t(e_t^2 - 2) + \mathcal{W}_{e_t}^3e_t((-446\eta^2 + 2144\eta - 4876)e_t^2 \right. \end{aligned}$$

$$+ (61(\eta - 4)\eta + 1445)e_t^4 + \eta(301\eta - 640) - 1399) \Big] , \quad (214)$$

$${}^0\tilde{\alpha}_{[22]} := -i\epsilon^2 \frac{12|E|e_t(e_t^2 - 2)}{\mathcal{W}_{e_t}} + i\epsilon^4 |E|^2 \left[\frac{6e_t((54\eta - 151)e_t^4 + (600 - 148\eta)e_t^2 - 4(8\eta + 37))}{7\mathcal{W}_{e_t}^3} + 12(5 - 2\eta)e_t \right] , \quad (215)$$

$${}^0\tilde{\alpha}_{[32]} := -i2\mathcal{W}_{e_t}^3 e_t - i\epsilon^2 \frac{1}{21}|E|\mathcal{W}_{e_t}e_t((15\eta + 149)e_t^2 + 153\eta - 667) + i\epsilon^4 \frac{|E|^2}{63\mathcal{W}_{e_t}} e_t \left[3024\mathcal{W}_{e_t}(2\eta - 5) + 740\eta^2 - 707\eta - 20795 + (-756\mathcal{W}_{e_t}(2\eta - 5) - 643\eta^2 + 1651\eta - 14042)e_t^2 + 5(\eta(31\eta - 112) + 443)e_t^4 \right] , \quad (216)$$

$${}^0\tilde{\alpha}_{[42]} := -i\frac{8}{21}|E|\mathcal{W}_{e_t}^3\epsilon^2(3\eta + 62)e_t + i\epsilon^4 \frac{|E|^2}{189}\mathcal{W}_{e_t}e_t \left[6804\mathcal{W}_{e_t}(2\eta - 5) + 1213\eta^2 + 15821\eta + 2126 + ((8627 - 701\eta)\eta - 37414)e_t^2 \right] , \quad (217)$$

$${}^0\tilde{\alpha}_{[52]} := i\frac{4}{7}|E|\mathcal{W}_{e_t}^5\epsilon^2(3\eta - 1)e_t + i\epsilon^4 \frac{|E|^2}{63}\mathcal{W}_{e_t}e_t(e_t^2 - 1) \left[3(4\eta(14\eta + 67) - 77)e_t^2 + 13\eta(31\eta - 25) + 12269 \right] , \quad (218)$$

$${}^0\tilde{\alpha}_{[62]} := -i\frac{|E|^2}{189}\mathcal{W}_{e_t}^5(\eta(3673\eta + 14153) - 10828)e_t , \quad (219)$$

$${}^0\tilde{\alpha}_{[72]} := i\epsilon^4 \frac{5}{63}|E|^2\mathcal{W}_{e_t}^7(4\eta(53\eta + 103) - 131)e_t , \quad (220)$$

$${}^{(2)}I^{21} = 0 , \quad (221)$$

$${}^{(2)}I^{20} = 16\sqrt{\frac{\pi}{15}}|E|\mu \left\{ \sum_{k=0}^5 {}^0\alpha_{[k0]}\mathcal{F}_{[k0]}(u) \right\} , \quad (222)$$

$${}^0\alpha_{[00]} := 1 - \frac{3}{14}|E|\epsilon^2(3\eta - 1) + \epsilon^4 \frac{1}{42}|E|^2(-11\eta^2 - 16\eta + 5) , \quad (223)$$

$${}^0\alpha_{[10]} := -1 + \frac{3}{14}|E|\epsilon^2(3\eta - 1) + \epsilon^4 |E|^2 \left[\frac{30 - 12\eta}{\mathcal{W}_{e_t}} + \frac{1}{42}(\eta(11\eta + 16) - 5) \right] , \quad (224)$$

$${}^0\alpha_{[20]} := \frac{2}{7}|E|\epsilon^2(\eta - 26) + \frac{|E|^2}{126\mathcal{W}_{e_t}^2} \left[((1877 - 323\eta)\eta - 3682)e_t^2 + 756\mathcal{W}_{e_t}(2\eta - 5) + 323\eta^2 + 1651\eta + 1666 \right] , \quad (225)$$

$${}^0\alpha_{[30]} := \frac{2}{7}|E|\epsilon^2(26 - \eta)\mathcal{W}_{e_t}^2 + \epsilon^4 \frac{1}{21}|E|^2[(686 - 2\eta(31\eta + 197))e_t^2 + \eta(41\eta + 625) - 1610] , \quad (226)$$

$${}^0\alpha_{[40]} := -\epsilon^4 \frac{1}{126}|E|^2\mathcal{W}_{e_t}^2(\eta(1517\eta + 7549) - 8645) , \quad (227)$$

$${}^0\alpha_{[50]} := +\epsilon^4 \frac{1}{42}|E|^2\mathcal{W}_{e_t}^4(4\eta(79\eta + 179) - 217) , \quad (228)$$

$${}^{(2)}S^{22} = 0 , \quad (229)$$

$${}^{(2)}S^{21} = \frac{32}{3}\sqrt{\frac{\pi}{5}}\sqrt{1 - e_t^2\eta}|E|^{3/2}(m_1 - m_2)e^{-i\mathcal{KM}}e^{-i\phi_0} \times$$

$$\left\{ \sum_{k=2}^5 {}^1\beta_{[k1]} \mathcal{F}_{[k1]}(u) + \sum_{k=3}^5 {}^1\tilde{\beta}_{[k1]} \mathcal{F}_{S[k1]}(u) \right\}, \quad (230)$$

$${}^1\beta_{[21]} := -\frac{|E|\epsilon^2 ((899 - 3695\eta)e_t^2 + 3751\eta - 955)}{28 \mathcal{W}_{e_t}^2} - 1, \quad (231)$$

$${}^1\beta_{[31]} := 1 - e_t^2 + |E|\epsilon^2 \left(\frac{1}{28} ((675 - 3639\eta)e_t^2 + 3695\eta - 1067) + \frac{1}{7} \mathcal{W}_{e_t}^2 (2077 - 2054\eta) \right), \quad (232)$$

$${}^1\beta_{[41]} := \frac{1}{14} |E| \mathcal{W}_{e_t}^2 \epsilon^2 (10340\eta - 10497), \quad (233)$$

$${}^1\beta_{[51]} := \frac{3}{14} |E| \mathcal{W}_{e_t}^4 \epsilon^2 (2133 - 2068\eta), \quad (234)$$

$${}^1\tilde{\beta}_{[31]} := -i \mathcal{W}_{e_t} e_t + \frac{i|E|\epsilon^2 e_t ((3639\eta - 675)e_t^2 - 3751\eta + 955)}{28 \mathcal{W}_{e_t}}, \quad (235)$$

$${}^1\tilde{\beta}_{[41]} := -\frac{1}{7} i |E| \mathcal{W}_{e_t} \epsilon^2 (2068\eta - 2049) e_t, \quad (236)$$

$${}^1\tilde{\beta}_{[51]} := \frac{3}{14} i |E| \mathcal{W}_{e_t}^3 \epsilon^2 (2068\eta - 2133) e_t, \quad (237)$$

$$S^{(2)20} = 0, \quad (238)$$

$$I^{(3)33} = -4i \sqrt{\frac{2\pi}{21}} \eta \delta_m \sqrt{|E|^3} \mathcal{W}_{e_t} e_t^{-3} e^{-i3\mathcal{K}\mathcal{M}} e^{-i3\phi_0} \times \left\{ \sum_{k=0}^7 {}^1\alpha_{[k3]} \mathcal{F}_{[k3]}(u) + \sum_{k=1}^7 {}^1\tilde{\alpha}_{[k3]} \mathcal{F}_{S[k3]}(u) \right\}, \quad (239)$$

$${}^1\alpha_{[03]} := -2e_t^2 + 8 - \frac{|E|\epsilon^2 ((5\eta - 7)e_t^4 + (191\eta - 829)e_t^2 - 268\eta + 1124)}{6 \mathcal{W}_{e_t}^2}, \quad (240)$$

$${}^1\alpha_{[13]} := 2(e_t^2 - 4) - \frac{|E|\epsilon^2 ((7 - 5\eta)e_t^4 + (1477 - 191\eta)e_t^2 + 4(67\eta - 497))}{6 \mathcal{W}_{e_t}^2}, \quad (241)$$

$${}^1\alpha_{[23]} := 7e_t^2 - 12 - \frac{|E|\epsilon^2 ((23\eta - 901)e_t^4 + (6305 - 2203\eta)e_t^2 + 2540\eta - 5764)}{36 \mathcal{W}_{e_t}^2}, \quad (242)$$

$${}^1\alpha_{[33]} := -e_t^2 (e_t^2 + 3) + 4 + \frac{1}{36} |E|\epsilon^2 (9(23\eta - 13)e_t^4 + (5353 - 395\eta)e_t^2 + 548\eta - 14668), \quad (243)$$

$${}^1\alpha_{[43]} := 20 \mathcal{W}_{e_t}^4 + \frac{1}{9} |E| \mathcal{W}_{e_t}^2 \epsilon^2 ((211\eta + 1162)e_t^2 + 3(383\eta - 745)), \quad (244)$$

$${}^1\alpha_{[53]} := -12 \mathcal{W}_{e_t}^6 - \frac{1}{9} |E| \mathcal{W}_{e_t}^4 \epsilon^2 (9(29\eta + 38)e_t^2 + 331\eta - 4973), \quad (245)$$

$${}^1\alpha_{[63]} = -\frac{4}{9} |E| \mathcal{W}_{e_t}^6 \epsilon^2 (140\eta + 467), \quad (246)$$

$${}^1\alpha_{[73]} = \frac{20}{3} |E| \mathcal{W}_{e_t}^8 \epsilon^2 (4\eta + 1), \quad (247)$$

$${}^1\tilde{\alpha}_{[13]} := -\frac{2ie_t (3e_t^2 - 4)}{\mathcal{W}_{e_t}} - \frac{i|E|\epsilon^2 e_t (3(19\eta - 89)e_t^2 - 268\eta + 1124)}{6 \mathcal{W}_{e_t}}, \quad (248)$$

$${}^1\tilde{\alpha}_{[23]} := |E|\epsilon^2 \left[\frac{36i(\eta - 4)e_t (3e_t^4 - 7e_t^2 + 4)}{\mathcal{W}_{e_t}^3} + \frac{36ie_t ((3\eta - 13)e_t^2 - 4(\eta - 5))}{\mathcal{W}_{e_t}} \right], \quad (249)$$

$$\begin{aligned}
{}^1\tilde{\alpha}_{[33]} &:= 24i \mathcal{W}_{e_t}^3 e_t - \frac{ie_t (29e_t^4 - 65e_t^2 + 36)}{\mathcal{W}_{e_t}} \\
&+ |E|\epsilon^2 \left[2i \mathcal{W}_{e_t}^3 (7 - 5\eta)e_t - \frac{30i(\eta - 4)e_t (3e_t^4 - 7e_t^2 + 4)}{\mathcal{W}_{e_t}^3} \right. \\
&\left. + \frac{ie_t (9e_t^2 ((35\eta + 23)e_t^2 - 247\eta + 653) + 2140\eta - 6836)}{36 \mathcal{W}_{e_t}} \right], \quad (250)
\end{aligned}$$

$$\begin{aligned}
{}^1\tilde{\alpha}_{[43]} &:= +16i \mathcal{W}_{e_t}^5 e_t - 8i \mathcal{W}_{e_t} e_t (2e_t^4 - 5e_t^2 + 3) + |E|\epsilon^2 \left[\frac{2}{3}i \mathcal{W}_{e_t}^5 (5\eta - 7)e_t \right. \\
&+ 2i \mathcal{W}_{e_t}^3 e_t ((5\eta - 7)e_t^2 + 139\eta - 497) + \frac{2}{3}i \mathcal{W}_{e_t} e_t (2(5\eta - 7)e_t^4 \\
&\left. + 37(11\eta - 40)e_t^2 - 505\eta + 1343) \right], \quad (251)
\end{aligned}$$

$$\begin{aligned}
{}^1\tilde{\alpha}_{[53]} &:= 12i \mathcal{W}_{e_t}^5 e_t \\
&+ |E|\epsilon^2 \left(48i \mathcal{W}_{e_t}^5 (\eta - 4)e_t + \frac{1}{3}i \mathcal{W}_{e_t}^3 e_t ((191\eta - 472)e_t^2 + 73\eta - 479) \right), \quad (252)
\end{aligned}$$

$${}^1\tilde{\alpha}_{[63]} = \frac{16}{9}i|E| \mathcal{W}_{e_t}^5 \epsilon^2 (20\eta + 113)e_t, \quad (253)$$

$${}^1\tilde{\alpha}_{[73]} = -\frac{20}{3}i|E| \mathcal{W}_{e_t}^7 \epsilon^2 (4\eta + 1)e_t, \quad (254)$$

$${}^{(3)}I^{32} = 0, \quad (255)$$

$$\begin{aligned}
{}^{(3)}I^{31} &= e^{-i\mathcal{KM}} e^{-i\phi_0} 8\sqrt{\frac{2\pi}{35}}|E|^{3/2} (m_1 - m_2) \eta \mathcal{W}_{e_t} \times \\
&\left\{ \sum_{k=0}^5 {}^1\alpha_{[k1]} \mathcal{F}_{[k1]}(u) + \sum_{k=1}^5 {}^1\tilde{\alpha}_{[k1]} \mathcal{F}_{S[k1]}(u) \right\}, \quad (256)
\end{aligned}$$

$${}^1\alpha_{[01]} := i - \frac{1}{12}i|E|\epsilon^2 \left(-\frac{24(\eta - 4)}{\mathcal{W}_{e_t}^2} + 5\eta - 7 \right), \quad (257)$$

$${}^1\alpha_{[11]} := -i - \frac{i|E|\epsilon^2 ((5\eta - 7)e_t^2 + 19\eta - 161)}{12 \mathcal{W}_{e_t}^2}, \quad (258)$$

$${}^1\alpha_{[21]} := -\frac{5i}{6} - \frac{5i|E|\epsilon^2 ((5\eta - 79)e_t^2 + 19\eta + 55)}{72 \mathcal{W}_{e_t}^2}, \quad (259)$$

$${}^1\alpha_{[31]} := \frac{5i \mathcal{W}_{e_t}^2}{6} - \frac{1}{24}i|E|\epsilon^2 ((13\eta + 1)e_t^2 - 53\eta + 279), \quad (260)$$

$${}^1\alpha_{[41]} := -\frac{1}{18}i|E| \mathcal{W}_{e_t}^2 \epsilon^2 (40\eta - 317), \quad (261)$$

$${}^1\alpha_{[51]} := \frac{1}{6}i|E| \mathcal{W}_{e_t}^4 \epsilon^2 (8\eta - 49), \quad (262)$$

$${}^1\tilde{\alpha}_{[11]} := -\frac{e_t}{\mathcal{W}_{e_t}} + \frac{|E| \mathcal{W}_{e_t} \epsilon^2 (89 - 19\eta)e_t}{12 \mathcal{W}_{e_t}^2}, \quad (263)$$

$${}^1\tilde{\alpha}_{[21]} := -\frac{6|E| \mathcal{W}_{e_t} \epsilon^2 e_t}{1 - e_t^2}, \quad (264)$$

$${}^1\tilde{\alpha}_{[31]} := \frac{5 \mathcal{W}_{e_t} e_t}{6} + \frac{|E|\epsilon^2 e_t (3(79 - 5\eta)e_t^2 + 95\eta - 157)}{72 \mathcal{W}_{e_t}}, \quad (265)$$

$${}^1\tilde{\alpha}_{[41]} := \frac{1}{9}|E|\mathcal{W}_{e_t}\epsilon^2(85-8\eta)e_t, \quad (266)$$

$${}^1\tilde{\alpha}_{[51]} := \frac{1}{6}|E|\epsilon^2(8\eta-49)e_t\mathcal{W}_{e_t}^3, \quad (267)$$

$$\begin{aligned} S^{(3)}_{32} &= e^{-i2\kappa\mathcal{M}}e^{-i2\phi_0}\frac{8}{3}\sqrt{\frac{2\pi}{7}}|E|^2\mathcal{W}_{e_t}^2\mu(1-3\eta)e_t^{-2}\times \\ &\quad \left\{ \sum_{k=2}^7 {}^2\beta_{[k2]}\mathcal{F}_{[k2]}(u) + \sum_{k=3}^7 {}^2\tilde{\beta}_{[k2]}\mathcal{F}_{S[k2]}(u) \right\}, \end{aligned} \quad (268)$$

$${}^2\beta_{[22]} := \frac{2i|E|\epsilon^2((\eta(10\eta-61)+20)e_t^2+(163-46\eta)\eta-50)}{3\mathcal{W}_{e_t}^2(3\eta-1)}-2i, \quad (269)$$

$$\begin{aligned} {}^2\beta_{[32]} &:= -2i\mathcal{W}_{e_t}^2 + \frac{2i|E|\epsilon^2}{45\mathcal{W}_{e_t}^2(3\eta-1)}\left[3(5\eta(16\eta-25)+28)e_t^4\right. \\ &\quad \left.+2(5\eta(73\eta-13)+4)e_t^2-5\eta(194\eta+223)+448\right], \end{aligned} \quad (270)$$

$${}^2\beta_{[42]} := 10i\mathcal{W}_{e_t}^2 + \frac{2i|E|\epsilon^2(3(5\eta(71\eta+370)-658)e_t^2+5\eta(662\eta-2477)+3809)}{45(3\eta-1)}, \quad (271)$$

$$\begin{aligned} {}^2\beta_{[52]} &:= -6i\mathcal{W}_{e_t}^4 \\ &\quad - \frac{2i|E|\mathcal{W}_{e_t}^2\epsilon^2(3(605\eta^2+910\eta-344)e_t^2-5\eta(230\eta+4147)+6847)}{45(3\eta-1)}, \end{aligned} \quad (272)$$

$${}^2\beta_{[62]} := -\frac{2i|E|\mathcal{W}_{e_t}^4\epsilon^2(4\eta(245\eta+604)-845)}{27\eta-9}, \quad (273)$$

$${}^2\beta_{[72]} := \frac{2i|E|\mathcal{W}_{e_t}^6\epsilon^2(20\eta(7\eta+8)-59)}{9\eta-3}, \quad (274)$$

$$\begin{aligned} {}^2\tilde{\beta}_{[32]} &:= -\frac{e_t(e_t^2-2)}{\mathcal{W}_{e_t}} \\ &\quad - \frac{|E|\epsilon^2e_t((\eta(8\eta-17)+4)e_t^4+3(\eta(22\eta-95)+30)e_t^2-92\eta^2+326\eta-100)}{3(3\eta-1)\mathcal{W}_{e_t}^3}, \end{aligned} \quad (275)$$

$$\begin{aligned} {}^2\tilde{\beta}_{[42]} &:= 4\mathcal{W}_{e_t}e_t \\ &\quad + \frac{2|E|\epsilon^2e_t((10(38-5\eta)\eta-157)e_t^2+1660\eta^2-1330\eta+302)}{45(3\eta-1)\mathcal{W}_{e_t}}, \end{aligned} \quad (276)$$

$$\begin{aligned} {}^2\tilde{\beta}_{[52]} &:= -6e_t\mathcal{W}_{e_t}^3 \\ &\quad + \frac{|E|\epsilon^2(-e_t\mathcal{W}_{e_t})(3(170\eta(\eta+2)-131)e_t^2+110\eta(10\eta-67)+2338)}{15(3\eta-1)}, \end{aligned} \quad (277)$$

$${}^2\tilde{\beta}_{[62]} := -\frac{8|E|\epsilon^2(4\eta(35\eta+121)-167)e_t\mathcal{W}_{e_t}^3}{27\eta-9}, \quad (278)$$

$${}^2\tilde{\beta}_{[72]} := \frac{2|E|\epsilon^2(20\eta(7\eta+8)-59)e_t\mathcal{W}_{e_t}^5}{9\eta-3}, \quad (279)$$

$$S^{(3)}_{30} = -16\sqrt{\frac{\pi}{105}}|E|^2e_t\sqrt{1-e_t^2}\mu(1-3\eta)\sum_3^5 {}^2\tilde{\beta}_{[k0]}\mathcal{F}_{S[k0]}(u), \quad (280)$$

$${}^2\tilde{\beta}_{[30]} := 1 - \frac{|E|\epsilon^2(((17-8\eta)\eta-4)e_t^2+(7-10\eta)\eta-2)}{3\mathcal{W}_{e_t}^2(3\eta-1)}, \quad (281)$$

$${}^2\tilde{\beta}_{[40]} := -\frac{2|E|\epsilon^2(2(\eta-121)\eta+79)}{9(3\eta-1)}, \quad (282)$$

$${}^2\tilde{\beta}_{[50]} := -\frac{|E|\mathcal{W}_{e_t}^2\epsilon^2(2(\eta-121)\eta+79)}{3-9\eta}, \quad (283)$$

$$\begin{aligned} I^{(4)}_{44} &= e^{-i4\mathcal{KM}} e^{-i4\phi_0} \frac{8}{3} \sqrt{\frac{2\pi}{7}} |E|^2 \mu(1-3\eta) e_t^{-4} \times \\ &\quad \left\{ \sum_{k=0}^9 {}^2\alpha_{[k4]} \mathcal{F}_{[k4]}(u) + \sum_{k=1}^9 {}^2\tilde{\alpha}_{[k4]} \mathcal{F}_{S[k4]}(u) \right\}, \end{aligned} \quad (284)$$

$$\begin{aligned} {}^2\alpha_{[04]} &:= e_t^4 - 8e_t^2 + 8 + \frac{|E|\epsilon^2}{55(3\eta-1)} \left[(5(41-8\eta)\eta-52)e_t^4 \right. \\ &\quad \left. - 8(5\eta(124\eta-531)+828)e_t^2 + 8(5\eta(256\eta-1103)+1708) \right], \end{aligned} \quad (285)$$

$$\begin{aligned} {}^2\alpha_{[14]} &:= -e_t^4 + 8e_t^2 - 8 + \frac{|E|\epsilon^2}{55(3\eta-1)} \left[(5\eta(8\eta-41)+52)e_t^4 \right. \\ &\quad \left. + 8(5\eta(124\eta-927)+1488)e_t^2 - 8(5\eta(256\eta-1895)+3028) \right], \end{aligned} \quad (286)$$

$$\begin{aligned} {}^2\alpha_{[24]} &:= \frac{1}{6} (-29e_t^4 + 112e_t^2 - 88) + \frac{|E|\epsilon^2}{330(3\eta-1)} \left[-5(\eta(314\eta-3185)+1121)e_t^4 \right. \\ &\quad \left. + 20(\eta(3829\eta-12717)+3903)e_t^2 - 8(5\eta(2921\eta-8542)+12788) \right], \end{aligned} \quad (287)$$

$$\begin{aligned} {}^2\alpha_{[34]} &:= \frac{1}{6} (3e_t^6 - 19e_t^4 + 24e_t^2 - 8) + \frac{|E|\epsilon^2}{330(3\eta-1)} \left[(7610\eta^2 - 73225\eta + 22598) e_t^4 \right. \\ &\quad \left. - 3(15\eta(58\eta-69)+218)e_t^6 + 40(\eta(411\eta+10226)-3407)e_t^2 \right. \\ &\quad \left. - 8(5\eta(536\eta+11245)-18912) \right], \end{aligned} \quad (288)$$

$$\begin{aligned} {}^2\alpha_{[44]} &:= -\frac{16}{3} \mathcal{W}_{e_t}^4 (e_t^2 - 6) + \frac{|E|\mathcal{W}_{e_t}^2\epsilon^2}{660(3\eta-1)} \left[(20\eta(214\eta-2359)+15563)e_t^4 \right. \\ &\quad \left. - 20(\eta(9705\eta-64529)+20707)e_t^2 + 8(5\eta(12953\eta-54357)+84244) \right], \end{aligned} \quad (289)$$

$$\begin{aligned} {}^2\alpha_{[54]} &:= -\frac{8}{3} \mathcal{W}_{e_t}^6 (3e_t^2 - 4) - \frac{|E|\mathcal{W}_{e_t}^4\epsilon^2}{660(3\eta-1)} \left[(60\eta(824\eta+67)-7371)e_t^4 \right. \\ &\quad \left. + 220(\eta(107\eta-363)+145)e_t^2 - 8(5\eta(7561\eta+45019)-77612) \right], \end{aligned} \quad (290)$$

$$\begin{aligned} {}^2\alpha_{[64]} &:= -\frac{140\mathcal{W}_{e_t}^8}{3} - \frac{|E|\mathcal{W}_{e_t}^6\epsilon^2}{99\eta-33} \left[(3\eta(4515\eta+11017)-12361)e_t^2 \right. \\ &\quad \left. + 4(\eta(9940\eta-26869)+8016) \right], \end{aligned} \quad (291)$$

$$\begin{aligned} {}^2\alpha_{[74]} &:= 20\mathcal{W}_{e_t}^{10} + \frac{|E|\mathcal{W}_{e_t}^8\epsilon^2}{99\eta-33} \left[3(15\eta(213\eta+95)-703)e_t^2 \right. \\ &\quad \left. - 4(\eta(535\eta+34698)-11300) \right], \end{aligned} \quad (292)$$

$${}^2\alpha_{[84]} := \frac{6|E|\mathcal{W}_{e_t}^{10}\epsilon^2(\eta(1225\eta+2427)-844)}{33\eta-11}, \quad (293)$$

$${}^2\alpha_{[94]} := -\frac{70|E|\mathcal{W}_{e_t}^{12}\epsilon^2(\eta(35\eta+9)-4)}{33\eta-11}, \quad (294)$$

$${}^2\tilde{\alpha}_{[14]} := \frac{4ie_t(e_t^4 - 3e_t^2 + 2)}{\mathcal{W}_{e_t}} + \frac{4i|E|\epsilon^2 e_t}{55(3\eta - 1)\mathcal{W}_{e_t}} \left[(5\eta(58\eta - 245) + 388)e_t^4 \right. \\ \left. + (5(2165 - 504\eta)\eta - 3364)e_t^2 + 10\eta(256\eta - 1103) + 3416 \right], \quad (295)$$

$${}^2\tilde{\alpha}_{[24]} := \frac{24i|E|\mathcal{W}_{e_t}\epsilon^2 e_t(e_t^4 - 8e_t^2 + 8)}{1 - e_t^2}, \quad (296)$$

$${}^2\tilde{\alpha}_{[34]} := \frac{ie_t(9e_t^6 - 55e_t^4 + 90e_t^2 - 44)}{3\mathcal{W}_{e_t}} + \frac{i|E|\epsilon^2 e_t}{330(3\eta - 1)\mathcal{W}_{e_t}} \left[15(\eta(133\eta - 815) + 273)e_t^6 \right. \\ \left. - 55(\eta(1009\eta - 4619) + 1457)e_t^4 + 4(5\eta(8286\eta - 37381) + 58391)e_t^2 \right. \\ \left. - 8(115\eta(127\eta - 578) + 20708) \right], \quad (297)$$

$${}^2\tilde{\alpha}_{[44]} := \frac{4}{3}ie_t\mathcal{W}_{e_t}^3(5e_t^2 - 12) + \frac{i|E|\mathcal{W}_{e_t}\epsilon^2 e_t}{330(3\eta - 1)} \left[20(4266\eta^2 - 3452\eta + 829)e_t^2 \right. \\ \left. + (1655(\eta - 1)\eta - 417)e_t^4 - 8(5\eta(3457\eta - 2049) + 1796) \right], \quad (298)$$

$${}^2\tilde{\alpha}_{[54]} := -2ie_t\mathcal{W}_{e_t}^5(e_t^2 - 8) + \frac{i|E|\epsilon^2 e_t\mathcal{W}_{e_t}^3}{220(3\eta - 1)} \left[(5\eta(581\eta - 1637) + 2473)e_t^4 \right. \\ \left. - 220(2\eta(73\eta - 520) + 333)e_t^2 + 88(915\eta^2 - 7615\eta + 2444) \right], \quad (299)$$

$${}^2\tilde{\alpha}_{[64]} := \frac{80}{3}ie_t\mathcal{W}_{e_t}^7 \\ + \frac{4i|E|\epsilon^2 e_t\mathcal{W}_{e_t}^5(11(6\eta(15\eta + 37) - 89)e_t^2 + 4(5\eta(340\eta - 131) + 76))}{99\eta - 33}, \quad (300)$$

$${}^2\tilde{\alpha}_{[74]} := -20ie_t\mathcal{W}_{e_t}^9 \\ - \frac{i|E|\epsilon^2 e_t\mathcal{W}_{e_t}^7(3(10\eta(197\eta + 111) - 563)e_t^2 + 4(\eta(3140\eta - 24249) + 7712))}{99\eta - 33}, \quad (301)$$

$${}^2\tilde{\alpha}_{[84]} := -\frac{4i|E|e_t\mathcal{W}_{e_t}^9\epsilon^2(\eta(1225\eta + 3483) - 1196)}{33\eta - 11}, \quad (302)$$

$${}^2\tilde{\alpha}_{[94]} := \frac{70i|E|e_t\mathcal{W}_{e_t}^{11}\epsilon^2(\eta(35\eta + 9) - 4)}{33\eta - 11}, \quad (303)$$

$$I^{42}_{(4)} = e^{-i2\mathcal{KM}} e^{-i2\phi_0} \frac{16}{21} \sqrt{2\pi} |E|^2 \mu (1 - 3\eta) e_t^{-2} \times \\ \left\{ \sum_{k=0}^7 {}^2\alpha_{[k2]} \mathcal{F}_{[k2]}(u) + \sum_{k=1}^7 {}^2\tilde{\alpha}_{[k2]} \mathcal{F}_{S[k2]}(u) \right\}, \quad (304)$$

$${}^2\alpha_{[02]} := e_t^2 - 2 + \frac{|E|\epsilon^2((5(41 - 8\eta)\eta - 52)e_t^2 - 2(5\eta(124\eta - 531) + 828))}{55(3\eta - 1)}, \quad (305)$$

$${}^2\alpha_{[12]} := 2 - e_t^2 + \frac{|E|\epsilon^2((5\eta(8\eta - 41) + 52)e_t^2 + 10\eta(124\eta - 927) + 2976)}{55(3\eta - 1)}, \quad (306)$$

$${}^2\alpha_{[22]} := \frac{8}{3} - \frac{11e_t^2}{6} + \frac{|E|\epsilon^2((632 - 5\eta(17\eta + 478))e_t^2 + 5\eta(2047\eta - 3411) + 4797)}{330(3\eta - 1)}, \quad (307)$$

$${}^2\alpha_{[32]} := \frac{1}{6}(-3e_t^4 + 11e_t^2 - 8) + \frac{|E|\epsilon^2}{330(3\eta - 1)} \left[165(1 - 4\eta)\eta e_t^4 \right. \\ \left. + (5\eta(1186\eta - 9227) + 14738)e_t^2 + 10(7369 - 527\eta)\eta - 23978 \right], \quad (308)$$

$${}^2\alpha_{[42]} := -\frac{10\mathcal{W}_{e_t}^4}{3} + \frac{|E|\mathcal{W}_{e_t}^2\epsilon^2((875\eta^2 - 6815\eta + 2738)e_t^2 - 5\eta(5715\eta + 3613) + 8167)}{660(3\eta - 1)}, \quad (309)$$

$${}^2\alpha_{[52]} := 2\mathcal{W}_{e_t}^6 - \frac{|E|\mathcal{W}_{e_t}^4\epsilon^2}{132(3\eta - 1)} \left[33(\eta(35\eta - 191) + 60)e_t^2 + (28933 - 6229\eta)\eta - 8915 \right], \quad (310)$$

$${}^2\alpha_{[62]} := -\frac{|E|\mathcal{W}_{e_t}^6\epsilon^2(\eta(4375\eta - 29191) + 8993)}{132(3\eta - 1)}, \quad (311)$$

$${}^2\alpha_{[72]} := \frac{|E|\mathcal{W}_{e_t}^8\epsilon^2(5\eta(125\eta - 653) + 983)}{44(3\eta - 1)}, \quad (312)$$

$${}^2\tilde{\alpha}_{[12]} := -2i\mathcal{W}_{e_t}e_t + \frac{2i|E|\epsilon^2e_t((5\eta(58\eta - 245) + 388)e_t^2 + 5(531 - 124\eta)\eta - 828)}{55(3\eta - 1)\mathcal{W}_{e_t}}, \quad (313)$$

$${}^2\tilde{\alpha}_{[22]} := \frac{12i|E|\epsilon^2e_t(e_t^2 - 2)}{\mathcal{W}_{e_t}}, \quad (314)$$

$${}^2\tilde{\alpha}_{[32]} := +\frac{ie_t(9e_t^4 - 25e_t^2 + 16)}{6\mathcal{W}_{e_t}} + \frac{i|E|\epsilon^2e_t}{660(3\eta - 1)\mathcal{W}_{e_t}} \left[15(\eta(133\eta - 815) + 273)e_t^4 + (5(13579 - 3569\eta)\eta - 21389)e_t^2 + 10\eta(2047\eta - 8163) + 25434 \right], \quad (315)$$

$${}^2\tilde{\alpha}_{[42]} := \frac{4}{3}ie_t\mathcal{W}_{e_t}^3 - \frac{i|E|\epsilon^2(-e_t\mathcal{W}_{e_t})}{660(3\eta - 1)} \left[(5\eta(355\eta - 4843) + 7527)e_t^2 + 10\eta(993\eta + 6575) - 22522 \right], \quad (316)$$

$${}^2\tilde{\alpha}_{[52]} := -2ie_t\mathcal{W}_{e_t}^5 + \frac{i|E|\epsilon^2e_t\mathcal{W}_{e_t}^3}{88(3\eta - 1)} \left[(\eta(145\eta - 937) + 337)e_t^2 - 22(\eta(113\eta - 289) + 87) \right], \quad (317)$$

$${}^2\tilde{\alpha}_{[62]} := \frac{i|E|\epsilon^2(\eta(625\eta - 4849) + 1511)e_t\mathcal{W}_{e_t}^5}{99\eta - 33}, \quad (318)$$

$${}^2\tilde{\alpha}_{[72]} := -\frac{i|E|\epsilon^2(5\eta(125\eta - 653) + 983)e_t\mathcal{W}_{e_t}^7}{44(3\eta - 1)}, \quad (319)$$

$$I^{(4)}_{40} = \frac{8}{21}\sqrt{\frac{\pi}{5}}|E|^2\mu(1 - 3\eta) \left\{ \sum_{k=0}^5 {}^2\alpha_{[k2]}\mathcal{F}_{[k2]}(u) \right\}, \quad (320)$$

$${}^2\alpha_{[00]} := 6 - \frac{6|E|\epsilon^2(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)}, \quad (321)$$

$${}^2\alpha_{[10]} := -6 + \frac{6|E|\epsilon^2(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)}, \quad (322)$$

$${}^2\alpha_{[20]} := -5 + \frac{|E|\epsilon^2(5\eta(82\eta - 1699) + 2711)}{55(3\eta - 1)}, \quad (323)$$

$${}^2\alpha_{[30]} := -5e_t^2 + 5 + \frac{|E|\mathcal{W}_{e_t}^2\epsilon^2(5\eta(2\eta + 163) - 218)}{55(3\eta - 1)}, \quad (324)$$

$${}^2\alpha_{[40]} := -\frac{3|E|\mathcal{W}_{e_t}^2\epsilon^2(20\eta(7\eta - 128) + 831)}{66\eta - 22}, \quad (325)$$

$${}^2\alpha_{[50]} := \frac{9|E|\mathcal{W}_{e_t}^4\epsilon^2(20\eta(7\eta-128)+831)}{110(3\eta-1)}, \quad (326)$$

$$S^{44}_{(4)} = 0, \quad (327)$$

$$S^{43}_{(4)} = e^{-i3\mathcal{KM}} e^{-i3\phi_0} \frac{16\sqrt{\frac{2\pi}{7}}|E|^{5/2}\mathcal{W}_{e_t}\delta_m(1-2\eta)\eta}{15e_t^3} \times \left\{ \sum_{k=2}^7 {}^3\beta_{[k0]}\mathcal{F}_{[k0]}(u) + \sum_{k=3}^7 {}^3\tilde{\beta}_{[k0]}\mathcal{F}_{S[k2]}(u) \right\}, \quad (328)$$

$${}^3\beta_{[23]} := 4 - 3e_t^2, \quad (329)$$

$${}^3\beta_{[33]} := 2(e_t^4 - 5e_t^2 + 4), \quad (330)$$

$${}^3\beta_{[43]} := \frac{1}{2}(-e_t^4 + 37e_t^2 - 36), \quad (331)$$

$${}^3\beta_{[53]} := \frac{1}{2}\mathcal{W}_{e_t}^4(33e_t^2 - 68), \quad (332)$$

$${}^3\beta_{[63]} := 70\mathcal{W}_{e_t}^6, \quad (333)$$

$${}^3\beta_{[73]} := -30\mathcal{W}_{e_t}^8, \quad (334)$$

$${}^3\tilde{\beta}_{[33]} := -i\mathcal{W}_{e_t}e_t(e_t^2 - 4), \quad (335)$$

$${}^3\tilde{\beta}_{[43]} := i\mathcal{W}_{e_t}e_t(12 - 7e_t^2), \quad (336)$$

$${}^3\tilde{\beta}_{[53]} := \frac{3}{2}i\mathcal{W}_{e_t}e_t(e_t^4 + 3e_t^2 - 4), \quad (337)$$

$${}^3\tilde{\beta}_{[63]} := -40ie_t\mathcal{W}_{e_t}^5, \quad (338)$$

$${}^3\tilde{\beta}_{[73]} := 30ie_t\mathcal{W}_{e_t}^7, \quad (339)$$

$$S^{42}_{(4)} = 0, \quad (340)$$

$$S^{41}_{(4)} = e^{-i\mathcal{KM}} e^{-i\phi_0} \frac{16\sqrt{2\pi}|E|^{5/2}\mathcal{W}_{e_t}\delta_m(1-2\eta)\eta}{35e_t} \times \left\{ \sum_{k=2}^5 {}^3\beta_{[k1]}\mathcal{F}_{[k0]}(u) + \sum_{k=3}^5 {}^3\tilde{\beta}_{[k1]}\mathcal{F}_{S[k2]}(u) \right\}, \quad (341)$$

$${}^3\beta_{[21]} := -1, \quad (342)$$

$${}^3\beta_{[31]} := -\frac{2\mathcal{W}_{e_t}^2}{3}, \quad (343)$$

$${}^3\beta_{[41]} := \frac{25\mathcal{W}_{e_t}^2}{6}, \quad (344)$$

$${}^3\beta_{[51]} := -\frac{5\mathcal{W}_{e_t}^4}{2}, \quad (345)$$

$${}^3\tilde{\beta}_{[31]} := -i\mathcal{W}_{e_t}e_t, \quad (346)$$

$${}^3\tilde{\beta}_{[41]} := -\frac{5}{3}i\mathcal{W}_{e_t}e_t, \quad (347)$$

$${}^3\tilde{\beta}_{[51]} := \frac{5}{2}ie_t\mathcal{W}_{e_t}^3, \quad (348)$$

$$S^{40}_{(4)} = 0, \quad (349)$$

$$I^{55}_{(5)} = e^{-i5\mathcal{KM}} e^{-i5\phi_0} \frac{8\sqrt{\frac{2\pi}{165}} |E|^{5/2} \delta_m (1-2\eta) \eta \mathcal{W}_{e_t}}{e_t^5} \times \left\{ \sum_{k=0}^9 {}^3\alpha_{[k5]} \mathcal{F}_{[k5]}(u) + \sum_{k=1}^9 {}^3\tilde{\alpha}_{[k5]} \mathcal{F}_{S[k5]}(u) \right\}, \quad (350)$$

$${}^3\alpha_{[05]} := i(e_t^4 - 12e_t^2 + 16), \quad (351)$$

$${}^3\alpha_{[15]} := -i(e_t^4 - 12e_t^2 + 16), \quad (352)$$

$${}^3\alpha_{[25]} := -\frac{1}{12}i(73e_t^4 - 396e_t^2 + 400), \quad (353)$$

$${}^3\alpha_{[35]} := -\frac{1}{6}i\mathcal{W}_{e_t}^2(3e_t^4 - 48e_t^2 + 80), \quad (354)$$

$${}^3\alpha_{[45]} := -\frac{1}{24}i(e_t - 1)(e_t + 1)(83e_t^4 - 1188e_t^2 + 1840), \quad (355)$$

$${}^3\alpha_{[55]} := \frac{1}{24}i\mathcal{W}_{e_t}^4(3e_t^4 - 1028e_t^2 + 2096), \quad (356)$$

$${}^3\alpha_{[65]} := -\frac{7}{6}i\mathcal{W}_{e_t}^6(23e_t^2 + 112), \quad (357)$$

$${}^3\alpha_{[75]} := \frac{95}{6}i\mathcal{W}_{e_t}^8(3e_t^2 - 8), \quad (358)$$

$${}^3\alpha_{[85]} := 210i\mathcal{W}_{e_t}^{10}, \quad (359)$$

$${}^3\alpha_{[95]} := -70i\mathcal{W}_{e_t}^{12}, \quad (360)$$

$${}^3\tilde{\alpha}_{[15]} := \frac{-5e_t^5 + 20e_t^3 - 16e_t}{\mathcal{W}_{e_t}}, \quad (361)$$

$${}^3\tilde{\alpha}_{[25]} := 0, \quad (362)$$

$${}^3\tilde{\alpha}_{[35]} := \frac{e_t(-42e_t^6 + 341e_t^4 - 692e_t^2 + 400)}{12\mathcal{W}_{e_t}}, \quad (363)$$

$${}^3\tilde{\alpha}_{[45]} := \frac{(e_t\mathcal{W}_{e_t})(107e_t^4 - 564e_t^2 + 560)}{12}, \quad (364)$$

$${}^3\tilde{\alpha}_{[55]} := -\frac{1}{8}e_t\mathcal{W}_{e_t}^3(19e_t^4 - 156e_t^2 + 240), \quad (365)$$

$${}^3\tilde{\alpha}_{[65]} := \frac{2}{3}e_t\mathcal{W}_{e_t}^5(31e_t^2 - 176), \quad (366)$$

$${}^3\tilde{\alpha}_{[75]} := \frac{5}{6}e_t\mathcal{W}_{e_t}^7(15e_t^2 + 16), \quad (367)$$

$${}^3\tilde{\alpha}_{[85]} := 140e_t\mathcal{W}_{e_t}^9, \quad (368)$$

$${}^3\tilde{\alpha}_{[95]} := -70e_t\mathcal{W}_{e_t}^{11}, \quad (369)$$

$$I^{54}_{(5)} = 0, \quad (370)$$

$$I^{53}_{(5)} = e^{-i3\mathcal{KM}} e^{-i3\phi_0} \frac{8\sqrt{\frac{2\pi}{33}} |E|^{5/2} \delta_m \eta (1-2\eta) \mathcal{W}_{e_t} (e_t^2 - 4)}{3e_t^3} \times \left\{ \sum_{k=0}^7 {}^3\alpha_{[k3]} \mathcal{F}_{[k3]}(u) + \sum_{k=1}^7 {}^3\tilde{\alpha}_{[k3]} \mathcal{F}_{S[k3]}(u) \right\}, \quad (371)$$

$${}^3\alpha_{[03]} := i, \quad (372)$$

$${}^3\alpha_{[13]} := -i, \quad (373)$$

$${}^3\alpha_{[23]} := -\frac{i(173e_t^2 - 404)}{60(e_t^2 - 4)}, \quad (374)$$

$${}^3\alpha_{[33]} := -\frac{i(9e_t^4 - 25e_t^2 + 16)}{30(e_t^2 - 4)}, \quad (375)$$

$${}^3\alpha_{[43]} := -\frac{7i(73e_t^4 - 301e_t^2 + 228)}{120(e_t^2 - 4)}, \quad (376)$$

$${}^3\alpha_{[53]} := \frac{7i\mathcal{W}_{e_t}^4(9e_t^2 - 4)}{120(e_t^2 - 4)}, \quad (377)$$

$${}^3\alpha_{[63]} := \frac{77i\mathcal{W}_{e_t}^6}{6(e_t^2 - 4)}, \quad (378)$$

$${}^3\alpha_{[73]} := -\frac{11i\mathcal{W}_{e_t}^8}{2(e_t^2 - 4)}, \quad (379)$$

$${}^3\tilde{\alpha}_{[13]} := \frac{4e_t - 3e_t^3}{\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (380)$$

$${}^3\tilde{\alpha}_{[23]} := 0, \quad (381)$$

$${}^3\tilde{\alpha}_{[33]} := -\frac{e_t(126e_t^4 - 495e_t^2 + 404)}{60\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (382)$$

$${}^3\tilde{\alpha}_{[43]} := -\frac{e_t(59e_t^4 - 183e_t^2 + 124)}{20\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (383)$$

$${}^3\tilde{\alpha}_{[53]} := -\frac{e_t\mathcal{W}_{e_t}^3(89e_t^2 - 284)}{40(e_t^2 - 4)}, \quad (384)$$

$${}^3\tilde{\alpha}_{[63]} := \frac{22e_t\mathcal{W}_{e_t}^5}{3(e_t^2 - 4)}, \quad (385)$$

$${}^3\tilde{\alpha}_{[73]} := -\frac{11e_t\mathcal{W}_{e_t}^7}{2(e_t^2 - 4)}, \quad (386)$$

$$I^{(5)52} = 0, \quad (387)$$

$$I^{(5)51} = e^{-i\kappa\mathcal{M}} e^{-i\phi_0} \frac{16\sqrt{\frac{\pi}{77}}|E|^{5/2}\mu\delta_m(1-2\eta)\mathcal{W}_{e_t}}{3me_t} \times \left\{ \sum_{k=0}^5 {}^3\alpha_{[k1]}\mathcal{F}_{[k1]}(u) + \sum_{k=1}^5 {}^3\tilde{\alpha}_{[k1]}\mathcal{F}_{S[k1]}(u) \right\}, \quad (388)$$

$${}^3\alpha_{[01]} := i, \quad (389)$$

$${}^3\alpha_{[11]} := -i, \quad (390)$$

$${}^3\alpha_{[21]} := -\frac{77i}{60}, \quad (391)$$

$${}^3\alpha_{[31]} := \frac{7i\mathcal{W}_{e_t}^2}{10}, \quad (392)$$

$${}^3\alpha_{[41]} := \frac{35i\mathcal{W}_{e_t}^2}{24}, \quad (393)$$

$${}^3\alpha_{[51]} := -\frac{7i\mathcal{W}_{e_t}^4}{8}, \quad (394)$$

$${}^3\tilde{\alpha}_{[11]} := -\frac{e_t}{\mathcal{W}_{e_t}}, \quad (395)$$

$${}^3\tilde{\alpha}_{[21]} := 0, \quad (396)$$

$${}^3\tilde{\alpha}_{[31]} := \frac{7e_t(11-6e_t^2)}{60\mathcal{W}_{e_t}}, \quad (397)$$

$${}^3\tilde{\alpha}_{[41]} := \frac{7}{12}e_t\mathcal{W}_{e_t}, \quad (398)$$

$${}^3\tilde{\alpha}_{[51]} := -\frac{7}{8}e_t\mathcal{W}_{e_t}^3, \quad (399)$$

$${}^{(5)}I^{50} = 0, \quad (400)$$

$${}^{(5)}S^{55} = 0, \quad (401)$$

$${}^{(5)}S^{54} = e^{-i4\mathcal{K}\mathcal{M}}e^{-i4\phi_0}\frac{64\sqrt{\frac{2\pi}{33}}|E|^3\mu(5(\eta-1)\eta+1)}{15e_t^4}\times$$

$$\left\{\sum_{k=2}^9{}^4\beta_{[k4]}\mathcal{F}_{[k4]}(u)+\sum_{k=3}^9{}^4\tilde{\beta}_{[k4]}\mathcal{F}_{S[k4]}(u)\right\}, \quad (402)$$

$${}^4\beta_{[24]} := i(e_t^4-3e_t^2+2), \quad (403)$$

$${}^4\beta_{[34]} := \frac{1}{6}i(-3e_t^6+31e_t^4-60e_t^2+32), \quad (404)$$

$${}^4\beta_{[44]} := -\frac{1}{12}i((e_t^2+3)(15e_t^2-34)e_t^2+76), \quad (405)$$

$${}^4\beta_{[54]} := \frac{1}{12}i\mathcal{W}_{e_t}^4(3e_t^4+218e_t^2-396), \quad (406)$$

$${}^4\beta_{[64]} := \frac{7}{6}i\mathcal{W}_{e_t}^6(19e_t^2+16), \quad (407)$$

$${}^4\beta_{[74]} := -\frac{5}{6}i\mathcal{W}_{e_t}^8(33e_t^2-100), \quad (408)$$

$${}^4\beta_{[84]} := -105i\mathcal{W}_{e_t}^{10}, \quad (409)$$

$${}^4\beta_{[94]} := 35i\mathcal{W}_{e_t}^{12}, \quad (410)$$

$${}^4\tilde{\beta}_{[34]} := -\frac{1}{4}e_t\mathcal{W}_{e_t}(e_t^4-8e_t^2+8), \quad (411)$$

$${}^4\tilde{\beta}_{[44]} := \frac{(-e_t\mathcal{W}_{e_t})(29e_t^4-112e_t^2+88)}{12}, \quad (412)$$

$${}^4\tilde{\beta}_{[54]} := -\frac{1}{8}e_t\mathcal{W}_{e_t}^3(3e_t^4-16e_t^2+8), \quad (413)$$

$${}^4\tilde{\beta}_{[64]} := -\frac{16}{3}e_t\mathcal{W}_{e_t}^5(e_t^2-6), \quad (414)$$

$${}^4\tilde{\beta}_{[74]} := -\frac{10}{3}e_t\mathcal{W}_{e_t}^7(3e_t^2-4), \quad (415)$$

$${}^4\tilde{\beta}_{[84]} := -70e_t\mathcal{W}_{e_t}^9, \quad (416)$$

$${}^4\tilde{\beta}_{[94]} := 35e_t\mathcal{W}_{e_t}^{11}, \quad (417)$$

$${}^{(5)}S^{53} = 0, \quad (418)$$

$$S^{(5)52} = e^{-i2\mathcal{K}\mathcal{M}} e^{-i2\phi_0} \frac{64\sqrt{\frac{2\pi}{11}}|E|^3\mu(5(\eta-1)\eta+1)\mathcal{W}_{e_t}^2}{45e_t^2} \times \left\{ \sum_{k=2}^7 {}^4\beta_{[k2]}\mathcal{F}_{[k2]}(u) + \sum_{k=3}^7 {}^4\tilde{\beta}_{[k2]}\mathcal{F}_{S[k2]}(u) \right\}, \quad (419)$$

$${}^4\beta_{[22]} := -i, \quad (420)$$

$${}^4\beta_{[32]} := \frac{1}{6}i(3e_t^2 - 10), \quad (421)$$

$${}^4\beta_{[42]} := -\frac{7}{12}i(3e_t^2 - 8), \quad (422)$$

$${}^4\beta_{[52]} := \frac{7}{12}i(3e_t^4 - 11e_t^2 + 8), \quad (423)$$

$${}^4\beta_{[62]} := -\frac{35i\mathcal{W}_{e_t}^4}{3}, \quad (424)$$

$${}^4\beta_{[72]} := 5i\mathcal{W}_{e_t}^6, \quad (425)$$

$${}^4\tilde{\beta}_{[32]} := -\frac{e_t(e_t^2 - 2)}{2\mathcal{W}_{e_t}}, \quad (426)$$

$${}^4\tilde{\beta}_{[42]} := -\frac{e_t(11e_t^2 - 16)}{6\mathcal{W}_{e_t}}, \quad (427)$$

$${}^4\tilde{\beta}_{[52]} := -\frac{e_t(3e_t^4 - 11e_t^2 + 8)}{4\mathcal{W}_{e_t}}, \quad (428)$$

$${}^4\tilde{\beta}_{[62]} := -\frac{20}{3}e_t\mathcal{W}_{e_t}^3, \quad (429)$$

$${}^4\tilde{\beta}_{[72]} := 5e_t\mathcal{W}_{e_t}^5, \quad (430)$$

$$S^{(5)51} = 0, \quad (431)$$

$$S^{(5)50} = \frac{32}{3}\sqrt{\frac{\pi}{1155}}|E|^3\mu\mathcal{W}_{e_t}(5(\eta-1)\eta+1)e_t \times \left\{ \sum_{k=3}^5 {}^4\tilde{\beta}_{[k0]}\mathcal{F}_{S[k0]}(u) \right\}, \quad (432)$$

$${}^4\tilde{\beta}_{[32]} := -1, \quad (433)$$

$${}^4\tilde{\beta}_{[42]} := -\frac{5}{3}, \quad (434)$$

$${}^4\tilde{\beta}_{[52]} := \frac{5\mathcal{W}_{e_t}^2}{2}, \quad (435)$$

$$I^{(6)66} = e^{-i6\mathcal{K}\mathcal{M}} e^{-i6\phi_0} \frac{32\sqrt{\frac{2\pi}{715}}|E|^3\mu(5(\eta-1)\eta+1)}{3e_t^6} \times \left\{ \sum_{k=0}^{11} {}^4\alpha_{[k6]}\mathcal{F}_{[k6]}(u) + \sum_{k=1}^{11} {}^4\tilde{\alpha}_{[k6]}\mathcal{F}_{S[k6]}(u) \right\}, \quad (436)$$

$${}^4\alpha_{[06]} := e_t^6 - 18e_t^4 + 48e_t^2 - 32, \quad (437)$$

$${}^4\alpha_{[16]} := -(e_t^2 - 2)(e_t^4 - 16e_t^2 + 16), \quad (438)$$

$${}^4\alpha_{[26]} := \frac{1}{60}(-437e_t^6 + 3666e_t^4 - 7536e_t^2 + 4384), \quad (439)$$

$${}^4\alpha_{[36]} := \frac{1}{30}(e_t - 1)(e_t + 1)(15e_t^6 - 444e_t^4 + 1648e_t^2 - 1408), \quad (440)$$

$${}^4\alpha_{[46]} := -\frac{1}{120} (e_t - 1) (e_t + 1) (283e_t^6 - 7086e_t^4 + 24208e_t^2 - 19680) , \quad (441)$$

$${}^4\alpha_{[56]} := \frac{1}{120} \mathcal{W}_{e_t}^4 (15e_t^6 - 5598e_t^4 + 33488e_t^2 - 37600) , \quad (442)$$

$${}^4\alpha_{[66]} := -\frac{1}{60} \mathcal{W}_{e_t}^6 (719e_t^4 - 1984e_t^2 - 12560) , \quad (443)$$

$${}^4\alpha_{[76]} := -\frac{1}{60} \mathcal{W}_{e_t}^8 (1605e_t^4 + 8336e_t^2 - 44496) , \quad (444)$$

$${}^4\alpha_{[86]} := -\frac{14}{15} \mathcal{W}_{e_t}^{10} (369e_t^2 + 406) , \quad (445)$$

$${}^4\alpha_{[96]} := \frac{14}{3} \mathcal{W}_{e_t}^{12} (51e_t^2 - 190) , \quad (446)$$

$${}^4\alpha_{[106]} := 924 \mathcal{W}_{e_t}^{14} , \quad (447)$$

$${}^4\alpha_{[116]} := -252 \mathcal{W}_{e_t}^{16} , \quad (448)$$

$${}^4\tilde{\alpha}_{[16]} := -2ie_t \mathcal{W}_{e_t} (3e_t^4 - 16e_t^2 + 16) , \quad (449)$$

$${}^4\tilde{\alpha}_{[26]} := 0 , \quad (450)$$

$${}^4\tilde{\alpha}_{[36]} := \frac{-i(e_t \mathcal{W}_{e_t}) (120e_t^6 - 1251e_t^4 + 3152e_t^2 - 2192)}{30} , \quad (451)$$

$${}^4\tilde{\alpha}_{[46]} := \frac{-i(e_t \mathcal{W}_{e_t}) (329e_t^6 - 2823e_t^4 + 5984e_t^2 - 3600)}{30} , \quad (452)$$

$${}^4\tilde{\alpha}_{[56]} := -\frac{-i(e_t \mathcal{W}_{e_t}^3) (55e_t^6 - 501e_t^4 + 1216e_t^2 - 880)}{20} , \quad (453)$$

$${}^4\tilde{\alpha}_{[66]} := \frac{-i(e_t \mathcal{W}_{e_t}^5) (301e_t^4 - 3296e_t^2 + 5360)}{15} , \quad (454)$$

$${}^4\tilde{\alpha}_{[76]} := \frac{i(e_t \mathcal{W}_{e_t}^7) (27e_t^4 + 704e_t^2 - 1776)}{12} , \quad (455)$$

$${}^4\tilde{\alpha}_{[86]} := -\frac{56i \mathcal{W}_{e_t}^7 e_t (2e_t^4 + 51e_t^2 - 53)}{5} , \quad (456)$$

$${}^4\tilde{\alpha}_{[96]} := \frac{28i \mathcal{W}_{e_t}^9 e_t (12e_t^4 - 35e_t^2 + 23)}{3} , \quad (457)$$

$${}^4\tilde{\alpha}_{[10 \ 6]} := -672i \mathcal{W}_{e_t}^{13} e_t , \quad (458)$$

$${}^4\tilde{\alpha}_{[11 \ 6]} := 252ie_t \mathcal{W}_{e_t}^{15} , \quad (459)$$

$${}^{(6)}I^{65} = 0 , \quad (460)$$

$${}^{(6)}I^{64} = e^{-i4\mathcal{K}\mathcal{M}} e^{-i4\phi_0} \frac{64\sqrt{\frac{\pi}{195}}|E|^3\mu(5(\eta-1)\eta+1)}{11e_t^4} \times \left\{ \sum_{k=0}^9 {}^4\alpha_{[k4]} \mathcal{F}_{[k4]}(u) + \sum_{k=1}^9 {}^4\tilde{\alpha}_{[k4]} \mathcal{F}_{S[k4]}(u) \right\} , \quad (461)$$

$${}^4\alpha_{[04]} := 8 - 8e_t^2 + e_t^4 , \quad (462)$$

$${}^4\alpha_{[14]} := -e_t^4 + 8e_t^2 - 8 , \quad (463)$$

$${}^4\alpha_{[24]} := \frac{1}{60} (-237e_t^4 + 1096e_t^2 - 936) , \quad (464)$$

$${}^4\alpha_{[34]} := \frac{1}{90} (-15e_t^6 - 97e_t^4 + 456e_t^2 - 344) , \quad (465)$$

$${}^4\alpha_{[44]} := -\frac{1}{360}(e_t - 1)(e_t + 1)(1629e_t^4 - 10888e_t^2 + 12584), \quad (466)$$

$${}^4\alpha_{[54]} := \frac{1}{360}\mathcal{W}_{e_t}^4(165e_t^4 - 4264e_t^2 + 9384), \quad (467)$$

$${}^4\alpha_{[64]} := \frac{7}{45}\mathcal{W}_{e_t}^6(31e_t^2 - 356), \quad (468)$$

$${}^4\alpha_{[74]} := \frac{1}{9}\mathcal{W}_{e_t}^8(57e_t^2 - 170), \quad (469)$$

$${}^4\alpha_{[84]} := 49\mathcal{W}_{e_t}^{10}, \quad (470)$$

$${}^4\alpha_{[94]} := -\frac{49\mathcal{W}_{e_t}^{12}}{3}, \quad (471)$$

$${}^4\tilde{\alpha}_{[14]} := -4ie_t\mathcal{W}_{e_t}(e_t^2 - 2), \quad (472)$$

$${}^4\tilde{\alpha}_{[24]} := 0, \quad (473)$$

$${}^4\tilde{\alpha}_{[34]} := -\frac{i(e_t\mathcal{W}_{e_t})(40e_t^4 - 217e_t^2 + 234)}{15}, \quad (474)$$

$${}^4\tilde{\alpha}_{[44]} := -\frac{i(e_t\mathcal{W}_{e_t})(209e_t^4 - 973e_t^2 + 874)}{45}, \quad (475)$$

$${}^4\tilde{\alpha}_{[54]} := \frac{i(e_t\mathcal{W}_{e_t}^3)(75e_t^4 - 431e_t^2 + 466)}{30}, \quad (476)$$

$${}^4\tilde{\alpha}_{[64]} := -\frac{2i(e_t\mathcal{W}_{e_t}^5)(251e_t^2 - 936)}{45}, \quad (477)$$

$${}^4\tilde{\alpha}_{[74]} := \frac{1}{18}i\mathcal{W}_{e_t}^7e_t(33e_t^2 - 248), \quad (478)$$

$${}^4\tilde{\alpha}_{[84]} := -\frac{98}{3}i\mathcal{W}_{e_t}^9e_t, \quad (479)$$

$${}^4\tilde{\alpha}_{[94]} := \frac{49}{3}ie_t\mathcal{W}_{e_t}^{11}, \quad (480)$$

$$I^{(6)}_{63} = 0, \quad (481)$$

$$I^{(6)}_{62} = e^{-i2\mathcal{K}\mathcal{M}}e^{-i2\phi_0}\frac{32\sqrt{\frac{2\pi}{13}}|E|^3\mu(5(\eta-1)\eta+1)}{33e_t^2}\times$$

$$\left\{\sum_{k=0}^7{}^4\alpha_{[k2]}\mathcal{F}_{[k2]}(u) + \sum_{k=1}^7{}^4\tilde{\alpha}_{[k2]}\mathcal{F}_{S[k2]}(u)\right\}, \quad (482)$$

$${}^4\alpha_{[02]} := e_t^2 - 2, \quad (483)$$

$${}^4\alpha_{[12]} := 2 - e_t^2, \quad (484)$$

$${}^4\alpha_{[22]} := \frac{97}{30} - \frac{39e_t^2}{20}, \quad (485)$$

$${}^4\alpha_{[32]} := \frac{1}{90}(-51e_t^4 + 95e_t^2 - 44), \quad (486)$$

$${}^4\alpha_{[42]} := -\frac{7}{72}(27e_t^4 - 89e_t^2 + 62), \quad (487)$$

$${}^4\alpha_{[52]} := -\frac{7}{360}\mathcal{W}_{e_t}^4(21e_t^2 - 26), \quad (488)$$

$${}^4\alpha_{[62]} := \frac{175\mathcal{W}_{e_t}^6}{36}, \quad (489)$$

$${}^4\alpha_{[72]} := -\frac{25 \mathcal{W}_{e_t}^8}{12}, \quad (490)$$

$${}^4\tilde{\alpha}_{[12]} := -2ie_t \mathcal{W}_{e_t}, \quad (491)$$

$${}^4\tilde{\alpha}_{[22]} := 0, \quad (492)$$

$${}^4\tilde{\alpha}_{[32]} := \frac{ie_t (40e_t^4 - 137e_t^2 + 97)}{30 \mathcal{W}_{e_t}}, \quad (493)$$

$${}^4\tilde{\alpha}_{[42]} := \frac{ie_t (137e_t^4 - 384e_t^2 + 247)}{90 \mathcal{W}_{e_t}}, \quad (494)$$

$${}^4\tilde{\alpha}_{[52]} := -\frac{i \mathcal{W}_{e_t} e_t (87e_t^4 - 284e_t^2 + 197)}{60}, \quad (495)$$

$${}^4\tilde{\alpha}_{[62]} := -\frac{25}{9}ie_t \mathcal{W}_{e_t}^5, \quad (496)$$

$${}^4\tilde{\alpha}_{[72]} := \frac{25}{12}ie_t \mathcal{W}_{e_t}^7, \quad (497)$$

$${}^{(6)}I^{61} = 0, \quad (498)$$

$${}^{(6)}I^{60} = \frac{64}{33} \sqrt{\frac{10\pi}{273}} |E|^3 \mu (5(\eta - 1)\eta + 1) \left\{ \sum_{k=0}^5 {}^4\alpha_{[k0]} \mathcal{F}_{[k0]}(u) \right\}, \quad (499)$$

$${}^4\alpha_{[00]} := 1, \quad (500)$$

$${}^4\alpha_{[10]} := -1, \quad (501)$$

$${}^4\alpha_{[20]} := -\frac{77}{60}, \quad (502)$$

$${}^4\alpha_{[30]} := \frac{7 \mathcal{W}_{e_t}^2}{10}, \quad (503)$$

$${}^4\alpha_{[40]} := \frac{35 \mathcal{W}_{e_t}^2}{24}, \quad (504)$$

$${}^4\alpha_{[50]} := -\frac{7 \mathcal{W}_{e_t}^4}{8}. \quad (505)$$

References

- [1] M. Tessmer and G. Schäfer, Phys. Rev. D **82**, 124064 (2010).
- [2] T. Damour and N. Deruelle, Ann. Inst. H. Poincaré A **43**, 107–132 (1985).
- [3] R. M. Memmesheimer, A. Gopakumar, and G. Schäfer, Phys. Rev. D **70**, 104011 (2004).
- [4] T. Damour, A. Gopakumar, and B. R. Iyer, Phys. Rev. D **70**, 064028 (2004).
- [5] A. Gopakumar and B. R. Iyer, Phys. Rev. D **65**, 084011 (2002).
- [6] N. J. Cornish and J. S. Key, Phys. Rev. D **82**, 044028 (2010).
- [7] A. Gopakumar, M. Hannam, S. Husa, and B. Brügmann, Phys. Rev. D **78**, 064026 (2008).
- [8] C. Reisswig, S. Husa, L. Rezzolla, E. N. Dorband, D. Pollney, and J. Seiler, Phys. Rev. D **80**, 124026 (2009).
- [9] P. Amaro-Seoane, C. Eichhorn, E. K. Porter, and R. Spurzem, Mon. Not. R. Astron. Soc. **401** (2010).
- [10] D. V. Gal'tsov, A. A. Matiukhin, and V. I. Petukhov, Phys. Lett. A **77**, 387 (1980).
- [11] C. Moreno-Garrido, J. Buitrago, and E. Mediavilla, Mon. Not. R. Astron. Soc. **266** (1994).
- [12] C. Moreno-Garrido, E. Mediavilla, and J. Buitrago, Mon. Not. R. Astron. Soc. **274**, 115–126 (1995).
- [13] V. Pierro, I. M. Pinto, E. Spallicci, E. Laserra, and F. Recano, Mon. Not. R. Astron. Soc. (325), 358–372 (2001).
- [14] N. Yunes, K. G. Arun, E. Berti, and C. M. Will, Phys. Rev. D **80**, 084001 (2009).

- [15] M. Tessmer, J. Hartung, and G. Schäfer, *Class. Quant. Grav.* **27**, 165005 (2010).
- [16] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, *Phys. Rev. D* **79**, 104023 (2009).
- [17] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299–339 (1980).
- [18] M. Turner and C. M. Will, *ApJ* **220** (1978).
- [19] M. Maggiore, *Gravitational Waves, Volume 1: Theory and Experiments* (Oxford University Press, Oxford, England, 2007).
- [20] W. Junker and G. Schäfer, *Mon. Not. R. Astron. Soc.* **254**, 146–164 (1992).
- [21] L. Blanchet, T. Damour, and B. R. Iyer, *Phys. Rev. D* **51**(5360-5386) (1995).
- [22] A. Gopakumar and B. R. Iyer, *Phys. Rev. D* **56**, 7708–7731 (1997).
- [23] L. Blanchet and T. Damour, *Phil. Trans. R. Soc. A* **320**, 379–430 (1986).
- [24] T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Rev. D* **77**, 064032 (2008).
- [25] L. Blanchet, *Proc. R. Soc. A* **409**, 383–399 (1987).
- [26] L. Blanchet, *Living Rev. Relativity* **9**, 4 (2006).
- [27] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, *Phys. Rev. D* **77**, 064035 (2008).
- [28] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, *Phys. Rev. D* **77**, 064034 (2008).
- [29] K. G. Arun, L. Blanchet, B. R. Iyer, and S. Sinha, *Phys. Rev. D* **80**, 124018 (2009).
- [30] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd edition (Cambridge University Press, Cambridge, 1980).
- [31] P. Colwell, *Solving Kepler’s equation over three centuries* (Willman-Bell, Inc., Richmond, VA 23235, 1993).
- [32] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, 1st edition (Cambridge University Press, Cambridge, 2010).
- [33] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, *Phys. Rev. D* **62**, 084036 (2000).
- [34] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, *Phys. Rev. D* **63**, 044023 (2001).
- [35] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, *Phys. Rev. D* **72**, 029902(E) (2005).
- [36] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and P. A. Sundararajan, *Phys. Rev. D* **71**, 084008 (2005).
- [37] G. Faye, L. Blanchet, and A. Buonanno, *Phys. Rev. D* **74**, 104033 (2006).
- [38] L. Blanchet, A. Buonanno, and G. Faye, *Phys. Rev. D* **74**, 104034 (2006).
- [39] L. Blanchet, A. Buonanno, and G. Faye, *Phys. Rev. D* **75**, 049903(E) (2007).
- [40] L. Blanchet, A. Buonanno, and G. Faye, *Phys. Rev. D* **81**, 089901(E) (2010).
- [41] R. A. Porto, A. Ross, and I. Z. Rothstein, *JCAP* **03**, 009 (2011).